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THE INITIAL-VALUE PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION

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CONTENTS

For the Korteweg–de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0,$$

existence, uniqueness, regularity and continuous dependence results are established for both the pure initial-value problem (posed on $-\infty < x < \infty$) and the periodic initial-value problem (posed on $0 \leqslant x \leqslant l$ with periodic initial data). The results are sharper than those obtained previously in that the solutions provided have the same number of L_2-derivatives as the initial data and these derivatives depend continuously on time, as elements of L_2. The same equation with dissipative and forcing terms added is also examined.

A by-product of the methods used is an exact relation between solutions of the Korteweg–de Vries equation and solutions of an alternative model equation recently

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studied by Benjamin, Bona & Mahony (1972). It is proven that in the long-wave limit under which these equations are generally derived, the solutions of the two models posed for the same initial data are the same.

In the process of carrying out this programme, new results are obtained for the latter model equation.

1. Introduction

The equation considered here was derived over 75 years ago by D. J. Korteweg and G. de Vries as a model for long water waves propagating in a channel. The significance of their ideas went more or less unappreciated for several decades. This can be traced in part to an inadequate description of their work by Lamb in his monumental treatise on hydrodynamics. The appearance of the same equation derived as a rudimentary model for waves in a number of diverse physical systems has awakened the interest of physicists and mathematicians. It is now generally understood that the Korteweg–de Vries equation, or other comparable model equations, can be expected to appear as describing to the first approximation the behaviour of unidirectional long waves in nonlinear dispersive media. An account of the general ingredients of the derivation of such model equations together with further references to their derivation in specific physical situations is presented in Benjamin, Bona & Mahony (1972, § 2). A wealth of additional material may be found in the review articles of Benjamin (1974), Jeffrey & Kakutani (1972), Miura (1974) and Scott, Chu & McLaughlin (1973).

Considerable stimulus to the study of the K.–dV. equation was provided by the research of a group at the Plasma Physics Laboratory of Princeton University. They built on earlier work of Whitham (1965) and Kruskal & Zabusky (1965), obtaining very interesting results, some of which are used here. An account of this work can be found in a series of papers collectively entitled 'Korteweg–de Vries equation and generalizations' which is referenced in the bibliography (cf. Gardner, Green, Kruskal & Miura 1967; Kruskal, Miura, Gardner & Zabusky 1970; Kruskal & Miura 1974; Miura, Gardner & Kruskal 1968; Gardner *et al.* 1974).

Rigorous existence theory for K.–dV. was begun by Sjöberg (1967) in an Uppsala University Computer Science Department report. For spatially periodic data with three L_2-derivatives Sjöberg (1970) proved existence of solutions to the K.–dV. equation which have three L_2-derivatives. For the same problem, Teman (1969) has shown existence of weak solutions corresponding to periodic initial data with one or two L_2-derivatives, while the later results of Tsutsumi & Mukasa (1971), when specialized to K.–dV., imply existence of solutions with m L_2-derivatives corresponding to initial data with m L_2-derivatives, $m \geqslant 1$. For the case of primary interest here, the pure initial-value problem posed on the entire real line, Kametaka (1969) has announced results in which the solutions had three less L_2-derivatives than the initial data. Again for the pure initial-value problem Tsutsumi, Mukasa & Iino (1970) have announced results (subsequently established by Tsutsumi & Mukasa 1971) for a generalized K.–dV. which when specialized to K.–dV. itself imply existence of solutions with m L_2-derivatives corresponding to initial data with m L_2-derivatives, $m \geqslant 1$. The top three spatial derivatives proven to exist may be discontinuous functions of time however. Dushane (1971, 1974) has discussed a related class of third order nonlinear evolution equations in which the solution proven to exist has the same number of L_2 derivatives as the initial data. He must assume his initial data has six L_2-derivatives before he obtains any solution at all and his results exclude the K.–dV. equation. Tsutsumi (1972) and Mukasa & Iino (1971) have also given existence theories for various generalizations of the basic K.–dV. equation. Menikoff (1972) has proved existence of unbounded solutions of K.–dV. His results are not strictly comparable to ours, and he does not make plain what function class his solutions lie in, but he appears to need conditions on the first seven derivatives of the initial data in order to establish existence of solutions. None of the above work gives consideration to the question of continuous dependence of solutions on the initial data.

The method of proof presented here is to regularize the K.–dV. equation by the addition of the linear term $-\epsilon u_{xxt}$, establish results for the regularized problem and then pass to the limit $\epsilon \downarrow 0$. It is inferred that if the initial data has $s \geqslant 2$ L_2-derivatives then corresponding to the given data there exists a unique solution (in the sense of distributions) which has $s \geqslant 2$ L_2-derivatives, all of which depend continuously on time. (In fact, as is shown in appendix A, existence can be proved even for initial data which is in L_2 and has only one L_2-derivative.) In particular, it is only required that the initial data has four L_2-derivatives in order to ensure existence of a classical solution to the problem (by which is meant a function for which all derivatives expressed in the differential equation exist and are continuous and which satisfies the equation pointwise). Against these positive results, however, we must set our failure to control the solution in C_b^k function classes.† Examination of the related linear problem

$$u_t + u_{xxx} = 0, \quad u(x, 0) = g(x)$$

seems to hold out little hope of being able to control the solution in C_b^k except by controlling $k+1$ L_2-derivatives (cf. Benjamin *et al.* 1972, p. 56). In this latter aspect, the K.–dV. equation is not as satisfactory mathematically as the alternative model equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.1}$$

proposed in Benjamin *et al.* (1972), which does allow direct control of solutions in C_b^k.

The regularization chosen may at first sight appear peculiar. Certainly the regularization $+\epsilon u_{xxxx}$, used first by Temam for K.–dV. and later by Dushane for a broad class of one dimensional evolution equations, appears to be more attractive in that it clearly guarantees good properties of solutions of the regularized equation. A price to be paid in using this regularization is that at least weak control of the fourth derivative must be established. Regularizing with a lower order term might well be preferable *provided* reasonable solutions for the regularized equation can be shown to exist. An examination of the 'dispersion relation' for the linear terms in our regularization seems to indicate 'on physical grounds' that smooth solutions should obtain (cf. Benjamin *et al.* 1972, § 2 for the definition and a discussion of dispersion in the context of one dimensional long-wave models). It turns out that this heuristic reasoning, based on consideration of the linear dispersion for the regularized problem, can be put on a firm mathematical base.

A bonus of proving results for K.–dV. by using the regularization suggested here is that the regularized equation can be used to throw some light on the question of whether K.–dV. or the equation (1.1) provides a 'better' model for long waves in nonlinear dispersive media. It is demonstrated that in the long-wave limit where amplitude is taken to be inversely proportional to the square of the wave length, which is the usual assumption made in deriving K.–dV., the solutions to K.–dV. and to (1.1) corresponding to the same initial data lie very close to each other (in a sense to be made precise, but which certainly implies the solutions are pointwise close

† $C_b^k = \{k$ times continuously differentiable functions which are bounded with their first k derivatives on $\mathbb{R}\}$.

together). This result seems to indicate that at least for the pure initial-value problem or the periodic initial-value problem posed for the purposes of modelling genuinely long waves, considerations other than the modelling properties should guide in choosing between K.–dV. and (1.1). For example, in computing numerically, (1.1) appears to be easier to stabilize for long waves of small amplitude (cf. the remarks by Benjamin *et al.* 1972, § 2, and the recent work of Hammack 1973 and Wahlbin 1974). Whereas for theoretical considerations, it might be convenient to have available the very considerable arsenal of formalism developed recently for K.–dV. (cf. Miura *et al.* 1968, Kruskal *et al.* 1970; Segur 1973).

The plan of the paper is as follows. In §§ 2–7 attention is restricted to the technically more challenging case of the pure initial-value problem posed on the entire real line. For the case of periodic initial values, defined on $[0, l]$ say with periodic boundary conditions imposed, the proofs are the same except some simplification is possible due to the boundedness of the underlying spatial domain. A discussion of the periodic case is included in appendix B. In §§ 2–7, an effort has been made to prove the results by using as little as we could of the techniques of modern functional analysis in hopes that the theory presented would be accessible to scientists interested in problems of wave propagation, but not well acquainted with the latest mathematical techniques. In appendix A, it is shown how an existence theory can be deduced rather more efficiently using weak compactness and interpolation ideas.

2. Statement of the problem and principal results

First remark that by changing to a set of coordinates moving with the wave, the initial-value problem for the Korteweg–de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{2.1}$$

can be put in the slightly simpler form

$$\left.\begin{aligned} u_t + uu_x + u_{xxx} &= 0, \\ u(x, 0) &= g(x), \end{aligned}\right\} \tag{2.2}$$

for $\quad t \geqslant 0, \quad -\infty < x < \infty.$

As usual, denote by $L_2 = L_2(\mathbb{R})$ the Hilbert space of measurable real-valued functions defined on $\mathbb{R}$ which are square integrable. Here $\mathbb{R}$ denotes the real line. For integers $s \geqslant 0$, let $H^s = H^s(\mathbb{R})$ be the Sobolev space

$$\{f \in L_2 : f^{(k)} = \mathrm{d}^k f/\mathrm{d}x^k \in L_2,\ 1 \leqslant k \leqslant s\},$$

normed in the standard way

$$\|f\|_s^2 = \sum_{j=0}^{s} \int_{-\infty}^{\infty} |f^{(j)}(x)|^2\,\mathrm{d}x. \tag{2.3}$$

Thus $H^0 = L_2$, and the norm in L_2 will be denoted simply by $\| \ \|$, omitting the subscript. By using Plancherel's theorem to express this norm in the Fourier transformed plane, there emerges the useful form

$$\|f\|_s^2 = \int_{-\infty}^{\infty} (1 + k^2 + \ldots + k^{2s})\,|\hat{f}(k)|^2\,\mathrm{d}k,$$

where $\hat{f}$ denotes the Fourier transform of f.

In order to describe the evolution of the spatial structure, the following Banach spaces are needed. For T a positive real number or $+\infty$, and non-negative integers s, the space $\mathscr{H}^s_T = C(0, T; H^s)$ consists of the functions u: $\mathbb{R} \times [0, T] \to \mathbb{R}$ which, for each $t \in [0, T]$, have

$u(\cdot,t)\in H^s$, and for which the mapping $u\colon[0,T]\to H^s$ is continuous and bounded. $\mathscr{H}^0_T$ will usually be written simply $\mathscr{H}_T$. The norm on $\mathscr{H}^s_T$ is

$$\|u\|_{\mathscr{H}^s_T} = |||u|||_s = \sup_{0\leqslant t\leqslant T}\|u(\cdot,t)\|_s.$$

For integers $k\geqslant 0$, there is likewise defined $C^k(0,T;H^s) = \mathscr{H}^{s,k}_T$ as the functions $u\in\mathscr{H}^s_T$ such that $\partial^j_t u\in\mathscr{H}^s_T$ for $0\leqslant j\leqslant k$. The norm in this case is

$$\|u\|_{\mathscr{H}^{s,k}_T} = |||u|||_{s,k} = \sup_{0\leqslant t\leqslant T}\sup_{0\leqslant j\leqslant k}\|\partial^j_t u(\cdot,t)\|_s.$$

Thus $\mathscr{H}^{s,0}_T = \mathscr{H}^s_T$. A few simple properties of these spaces are summarized in the following proposition.

Proposition 1. For integers $s\geqslant 1$ and $k\geqslant 0$ the following results hold.

(*a*) $f\in H^s \Rightarrow f, f', \ldots, f^{(s-1)}$ are bounded uniformly continuous functions which converge to 0 at $\pm\infty$.

(*b*) $f,g\in H^s\Rightarrow f\cdot g\in H^s$.

(*c*) $u\in\mathscr{H}^{s,k}_T\Rightarrow \partial^j_x\partial^l_t u$ is a bounded continuous function on $\mathbb{R}\times[0,T]$ (uniformly continuous on bounded time intervals) which converges to 0 as $x\to+\infty$, uniformly for bounded time intervals, for $0\leqslant j\leqslant s-1$, $0\leqslant l\leqslant k$.

(*d*) $u,v\in\mathscr{H}^{s,k}_T\Rightarrow u\cdot v\in\mathscr{H}^{s,k}_T$.

Remarks. Properties (*a*) and (*b*) above are standard results which can be found in many text-books.† Properties (*c*) and (*d*) are straightforward generalizations of (*a*) and (*b*).

Notation. Throughout the remainder of this paper the Sobolev norms of a function u of both spatial and temporal variables will always be applied to the spatial variables. Because of this uniformity, the simpler but less precise notation $\|u\|_s$ will be used instead of $\|u(\cdot,t)\|_s$ throughout. An analogous convention will be used regarding the $L_\infty(\mathbb{R})$ norm of a function u of both spatial and temporal variables.

It is obvious from the differential equation that differentiating a solution with respect to time reduces the regularity in the spatial variable by three x-derivatives. Hence it is natural to define the following space of functions. For integers $s\geqslant 0$,

$$\mathscr{X}_{s,T} = \mathscr{H}^s_T\cap\mathscr{H}^{s-3,1}_T\cap\mathscr{H}^{s-6,2}_T\cap\ldots. \tag{2.4}$$

That is, $\mathscr{X}_{s,T} = \{u\in\mathscr{H}^s_T\colon \partial^l_t u\in\mathscr{H}^{s-3l}_T \text{ for } l \text{ such that } s-3l\geqslant 0\}.$

With this notation in hand, the principal results for K.–dV. can be stated. These serve to define the aims of §§ 3–6 and appendix A. Results corresponding to weaker assumptions on the initial data are also given in appendix A.

Theorem. Let $g\in H^s$, where $s\geqslant 2$. Then there exists a unique solution $u\in\mathscr{X}_{s,\infty}$ to the initial-value problem (2.2) which depends continuously on the initial data.

In the above result, if $s<3$, then the term 'solution' connotes a solution in the sense of distributions. If $s=3$, then all the relevant derivatives exist almost everywhere and the equation is satisfied pointwise almost everywhere. If $s>3$, the derivatives expressed in the K.–dV. equation all exist and are continuous, and the equation is satisfied identically. That is, the solution is a classical solution of the initial-value problem. The continuous dependence result alluded to above will be spelled out more precisely in § 6.

† Cf. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series No. 30, Princeton University Press, Princeton, 1970.

3. Existence of solutions of the regularized initial-value problem

Consideration will be temporarily given to the initial-value problem

$$\left.\begin{aligned} u_t + uu_x + u_{xxx} - \epsilon u_{xxt} &= 0, \\ u(x,0) &= g(x), \end{aligned}\right\} \tag{3.1}$$

for $x \in \mathbb{R}$, $t \geqslant 0$ and fixed ϵ in the range $1 \geqslant \epsilon > 0$ say. Changing the dependent and independent variables as follows

$$v(x,t) = \epsilon u(\epsilon^{\frac{1}{2}}(x-t),\ \epsilon^{\frac{3}{2}}t), \tag{3.2}$$

transforms the problem (3.1) into the initial-value problem

$$\left.\begin{aligned} v_t + v_x + vv_x - v_{xxt} &= 0, \\ v(x,0) = h(x) &= \epsilon g(\epsilon^{\frac{1}{2}}x), \end{aligned}\right\} \tag{3.3}$$

for $x \in \mathbb{R}$ and $t \geqslant 0$. For ϵ fixed and positive, h is of the same function class as g and hence the existence and regularity theory for the initial-value problem (3.3) developed by Benjamin *et al.* (1972, § 3) may be used to advantage. The result is stated first in terms of the model system (3.3). The statement given here, which takes account of the L_2-properties of derivatives of order higher than one, provides a slight extension of the results derived in the last quoted reference. The essentials of the proof of this extension are included below.

Lemma 1. Suppose $h \in H^m$ where $m \geqslant 2$. Then there exists a unique solution u of the initial-value problem (3.3) which is in $\mathscr{H}^m_T$, for all finite $T > 0$. Furthermore, for $j \geqslant 1$, $\partial_t^j u \in \mathscr{H}^{m+1}_T$, for all finite $T > 0$.

Remarks. If $m \geqslant 2$, each term in the differential equation in (3.3) is a continuous function of x and t, and u satisfies the equation pointwise. That is, u is a classical solution to the initial-value problem. For $m = 1$, the result still holds, but u only satisfies the differential equation, for each t, pointwise for almost every x.

Proof. Since $m \geqslant 2$, the hypotheses are sufficient to imply the results of theorem 1 of § 3 and theorem 4 of § 4 of Benjamin *et al.* (1972). We may certainly conclude that (3.3) has a unique solution u which in particular lies in the function class $\mathscr{H}^1_\infty$. Further, as in Benjamin *et al.* (1972, § 3), u satisfies the integral equation derived from (3.3) by inverting $I - \partial_x^2$, integrating by parts, and then integrating up in time:

$$u(x,t) = h(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\,\{u(y,\tau) + \tfrac{1}{2}u(y,\tau)^2\}\,dy\,d\tau, \tag{3.4}$$

where $$K(z) = \tfrac{1}{2}\,\mathrm{sgn}\,(z)\,e^{-|z|}.$$

From this representation of u, it follows by induction that, for any finite $T > 0$, $u \in \mathscr{H}^m_T$. Assuming that $u \in \mathscr{H}^j_T$ for some j with $m > j \geqslant 1$, one sees from the representation (3.4) that

$$\partial_x^{j+1} u = h^{(j+1)} + \int_0^t \left[\partial_x^j(u + \tfrac{1}{2}u^2) - \int_{-\infty}^{\infty} \tfrac{1}{2}\,e^{-|x-y|}\,\partial_y^j(u + \tfrac{1}{2}u^2)\,dy\right] d\tau. \tag{3.5}$$

The right hand side of (3.5) is in $\mathscr{H}_T$ since u, and hence u^2 by proposition 1, is in $\mathscr{H}^j_T$. Thus $u \in \mathscr{H}^j_T$ and $\partial_x^{j+1} u \in \mathscr{H}_T$, whence $u \in \mathscr{H}^{j+1}_T$. Then $u \in \mathscr{H}^m_T$, and the induction cannot proceed further since h cannot be differentiated further. Note that if $m > 1$, we do not assert $u \in \mathscr{H}^m_\infty$ since the relations derived do not allow us to bound the growth in time of derivatives of order higher than one uniformly for all time.

If (3.4) is differentiated with respect to time, it appears that

$$u_t = \int_{-\infty}^{\infty} K(x-y)\,(u+\tfrac{1}{2}u^2)\,\mathrm{d}y. \tag{3.6}$$

An easy calculation from the Fourier transform shows that convolution with K maps H^m continuously to H^{m+1}, and so $\mathscr{H}_T^m$ to $\mathscr{H}_T^{m+1}$. Thus it follows that $u_t \in \mathscr{H}_T^{m+1}$. We finish by proceeding inductively from (3.6). Assuming that $\partial_t^j u \in \mathscr{H}_T^{m+1}$ for $j \leqslant s$, where $s \geqslant 1$, write

$$\partial_t^{s+1} u = \int_{-\infty}^{\infty} K(x-y)\,\partial_t^s(u+\tfrac{1}{2}u^2)\,\mathrm{d}y.$$

By proposition 1, $\partial_t^s(u+\frac{1}{2}u^2) \in \mathscr{H}_T^m$. Hence the mapping properties of convolution with K alluded to above allow the conclusion $\partial_t^{s+1}u \in \mathscr{H}_T^{m+1}$. The proof of the lemma is now complete.

The second lemma is a corollary to this result and the transformation (3.2) between the two initial-value problems (3.1) and (3.3).

Lemma 2. Suppose $g \in H^m$ where $m \geqslant 2$. Then there exists a unique solution u to the regularized K.-dV. equation (3.1) with initial value g which lies in $\mathscr{H}_T^m$ for any finite $T > 0$. Furthermore, for $0 \leqslant l \leqslant m$, $\partial_t^l u \in \mathscr{H}_T^{m-l}$ for any finite $T > 0$.

Corollary. Let g be a C^∞ function all of whose derivatives are in L_2. Then there exists a unique C^∞ solution u to the regularized K.-dV. equation (3.1) which, with all its derivatives, lies in $\mathscr{H}_T$ for any finite $T > 0$.

4. Derivation of *a priori* estimates for solutions of the regularized initial-value problem

Roughly speaking, the last result of § 3 shows that the regularized K.-dV. equation has smooth solutions corresponding to smooth initial data. As is not uncommon when dealing with partial differential equations, the heart of the theory lies in the derivation of *a priori* bounds which smooth solutions must satisfy. The derivation of these bounds will be undertaken in the present section. Use is made of the first three non-trivial invariants for K.-dV. discovered by Miura *et al.* (1968), adapted to the regularized problem, and of some integral inequalities for solutions of the regularized K.-dV. equation. Note, incidentally, that we cannot avail ourselves of the bounds for the problem (3.3) because the inverse of the transformation (3.2) becomes singular at $\epsilon = 0$. Hence bounds must be derived directly in terms of the regularized K.-dV. equation (3.1).

Throughout this section, the initial data g for (3.1) is assumed to be a C^∞ function all of whose derivatives are in L_2. The set of all such functions will be denoted H^∞. Then u will denote the unique solution of (3.1) for the initial data g guaranteed by the corollary to lemma 2. By the corollary to lemma 2, u and all its spatial and temporal derivatives lie in $\mathscr{H}_T$ for all finite $T > 0$. The ϵ in (3.1) is still restricted to the range $0 < \epsilon \leqslant 1$.

Proposition 2. The solution u of (3.1) corresponding to g given in H^∞ satisfies the inequality

$$\|u\|_1 \leqslant a(\|g\|_1), \tag{4.1}$$

for all $t > 0$, independently of ϵ in $(0, 1]$, where $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing with $a(0) = 0$.

Proof. Multiply the regularized equation (3.1) by u and integrate over $\mathbb{R}$ and over $[0, t]$. After appropriate integrations by parts and allowing for the fact that u and all its partial derivatives tend to 0 at $\pm\infty$, there obtains, for all $t \geq 0$,

$$\int_{-\infty}^{\infty} [u^2(x,t) + \epsilon u_x^2(x,t)]\,\mathrm{d}x = \int_{-\infty}^{\infty} [g(x)^2 + \epsilon g'(x)^2]\,\mathrm{d}x. \tag{4.2}$$

Next multiply the regularized K.–dV. equation by $u^2 + 2u_{xx}$ and again integrate over $\mathbb{R}$ and over $[0, t]$. Making use of the identity

$$\frac{1}{2}\int_{-\infty}^{\infty} u^2 u_{xxt}\,\mathrm{d}x = -\int_{-\infty}^{\infty} u_{xx}u_{xxt}\,\mathrm{d}x$$

derived by multiplying the regularized K.–dV. equation by u_{xt} and integrating by parts, one finds that for all $t \geq 0$,

$$\int_{-\infty}^{\infty} [u_x^2 - \tfrac{1}{3}u^3]\,\mathrm{d}x = \int_{-\infty}^{\infty} [g'(x)^2 - \tfrac{1}{3}g(x)^3]\,\mathrm{d}x. \tag{4.3}$$

The relations (4.2) and (4.3) combine to give the desired result. For from (4.2) it follows that, independently of ϵ in $(0, 1]$,

$$\|u\| \leq \|g\|_1. \tag{4.4}$$

Then by using the elementary† inequality

$$\sup_{x\in\mathbb{R}} |f(x)| \leq (\|f\|\,\|f'\|)^{\frac{1}{2}} \leq \|f\|_1 \tag{4.5}$$

for $f \in H^1$, there is derived from (4.3) and (4.4) the inequality

$$\begin{aligned}
\int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x &= \frac{1}{3}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x + \int_{-\infty}^{\infty} [g'(x)^2 - \tfrac{1}{3}g(x)^3]\,\mathrm{d}x \\
&\leq \tfrac{1}{3}\sup_{x\in\mathbb{R}}|u| \int_{-\infty}^{\infty} u^2\,\mathrm{d}x + \int_{-\infty}^{\infty} g'(x)^2\,\mathrm{d}x + \tfrac{1}{3}\sup_{x\in\mathbb{R}}|g| \int_{-\infty}^{\infty} g^2\,\mathrm{d}x \\
&\leq \tfrac{1}{3}\|u\|_1\|u\|^2 + \|g'\|^2 + \tfrac{1}{3}\|g\|_1\|g\|^2 \\
&\leq \tfrac{1}{3}\|u\|_1\|g\|_1^2 + \|g\|_1^2 + \tfrac{1}{3}\|g\|_1^3.
\end{aligned}$$

Hence independently of $t \geq 0$ and ϵ in $(0, 1]$,

$$\|u\|_1^2 = \int_{-\infty}^{\infty} \{u^2 + u_x^2\}\,\mathrm{d}x \leq \tfrac{1}{3}\|u\|_1\|g\|_1^2 + (2\|g\|_1^2 + \tfrac{1}{3}\|g\|_1^3),$$

from which there follows immediately, upon solving the quadratic inequality in $\|u\|_1$ above, that independently of $t \geq 0$ and ϵ in $(0, 1]$

$$\|u\|_1 \leq a(\|g\|_1),$$

where $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing with $a(0) = 0$.

The bounds derived in proposition 2 above are sufficient to conclude existence of weak solutions of K.–dV., using arguments given in appendix A. Further *a priori* estimates are required, however, to establish existence of smooth solutions and in the task of comparing the K.–dV. equation with the model equation (1.1).

† The first inequality follows from representing $f^2(x) = \int_{-\infty}^{x} ff' - \int_{x}^{\infty} ff'$, bounding the integrals by $\int_{-\infty}^{\infty} |f|\,|f'|$ and then applying the Cauchy–Schwarz inequality. The second inequality follows from the trivial relation $ab \leq a^2 + b^2$. (The factor of $\frac{1}{2}$ is eschewed for the sake of tidiness.)

As will appear below, the next stage in the derivation of *a priori* bounds is the most difficult. Here is a simple and useful corollary of proposition 2 which follows immediately from (4.5).

Corollary. The solution u of the regularized K.-dV. initial-value problem (3.1) corresponding to a given $g \in H^\infty$ satisfies

$$\sup_{x\in\mathbb{R},\, t\geqslant 0} |u(x,t)| \leqslant a(\|g\|_1), \tag{4.6}$$

where a is the function in (4.1) of proposition 2.

Proposition 3. Let $T>0$ and $g\in H^\infty$ be given. Then there exists $\epsilon_0 = \epsilon_0(T, \|g\|_3)$ such that the solution u of (3.1) corresponding to g and any ϵ in the range $0<\epsilon\leqslant\epsilon_0$ satisfies the inequality

$$\|u\|_2 \leqslant a_1(\|g\|_3) \tag{4.7}$$

independently of t in $[0,T]$, where $a_1\colon \mathbb{R}^+\to\mathbb{R}^+$ is continuous monotone increasing and $a_1(0)=0$.

Proof. A lengthy calculation (multiply the regularized equation by $u^3+3u_x^2+6uu_{xx}+\frac{18}{5}u_{xxxx}$) gives the following identity.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}[\tfrac{9}{5}u_{xx}^2-3uu_x^2+\tfrac{1}{4}u^4]\,\mathrm{d}x = \epsilon\int_{-\infty}^{\infty}[u^3+3u_x^2+6uu_{xx}+\tfrac{18}{5}u_{xxxx}]\,u_{xxt}\,\mathrm{d}x.$$

This can be put in more illuminating form by a few integrations by parts.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}[(\tfrac{9}{5}-3\epsilon u)\,u_{xx}^2-3uu_x^2+\tfrac{1}{4}u^4+\tfrac{9}{5}\epsilon u_{xxx}^2]\,\mathrm{d}x = -\epsilon\int_{-\infty}^{\infty}[3u_tu_{xx}^2+3u^2u_xu_{xt}+6u_xu_{xx}u_{xt}]\,\mathrm{d}x. \tag{4.8}$$

Hence define $$V(t)=\int_{-\infty}^{\infty}[(\tfrac{9}{5}-3\epsilon u)\,u_{xx}^2-3uu_x^2+\tfrac{1}{4}u^4+\tfrac{9}{5}\epsilon u_{xxx}^2]\,\mathrm{d}x,$$

and integrate (4.8) over $[0,t]$ to obtain

$$V(t) = V(0)-\epsilon\int_0^t\int_{-\infty}^{\infty}[3u_tu_{xx}^2+3u^2u_xu_{xt}+6u_xu_{xx}u_{xt}]\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.9}$$

From the corollary to proposition 2, u is bounded for all x and t in terms of $\|g\|_1$ as in (4.6). Thus there is an $\epsilon_1>0$ such that if $0<\epsilon\leqslant\epsilon_1$, then

$$\tfrac{13}{5}\geqslant\tfrac{9}{5}-3\epsilon u\geqslant 1. \tag{4.10}$$

So in the range $\epsilon_1\geqslant\epsilon>0$, the identity (4.9) yields the following integral inequality.

$$\begin{aligned}\int_{-\infty}^{\infty}u_{xx}^2\,\mathrm{d}x &\leqslant \int_{-\infty}^{\infty}[(\tfrac{9}{5}-3\epsilon u)\,u_{xx}^2+\tfrac{1}{4}u^4+\tfrac{9}{5}\epsilon u_{xxx}^2]\,\mathrm{d}x\\ &\leqslant V(0)+3\int_{-\infty}^{\infty}(|u|\,|u_x|^2)\,\mathrm{d}x\\ &\quad+\epsilon\int_0^t\int_{-\infty}^{\infty}|3u_tu_{xx}^2+3u^2u_xu_{xt}+6u_xu_{xx}u_{xt}|\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{4.11}$$

The first two terms on the right hand side of (4.11) can be bounded, independently of ϵ in $(0,\epsilon_1]$, in terms of the H^3 norm of the initial data g as follows. If $\epsilon\leqslant\epsilon_1$, then by using (4.10) at $t=0$ and (4.5) twice,

$$\begin{aligned}V(0) &= \int_{-\infty}^{\infty}[(\tfrac{9}{5}-3\epsilon g)\,g''^2-3gg'^2+\tfrac{1}{4}g^4+\tfrac{9}{5}\epsilon g'''^2]\,\mathrm{d}x\\ &\leqslant\int_{-\infty}^{\infty}[\tfrac{13}{5}g''^2+3\,\|g\|_1\,g'^2+\tfrac{1}{4}\|g\|_1^2\,g^2+\tfrac{9}{5}\epsilon g'''^2]\,\mathrm{d}x\\ &\leqslant\tfrac{13}{5}\|g\|_2^2+3\|g\|_1^3+\tfrac{1}{4}\|g\|_1^4+\tfrac{9}{5}\epsilon_1\|g\|_3^2.\end{aligned} \tag{4.12}$$

Similarly, again making use of (4.5),

$$3\int_{-\infty}^{\infty}(|u|\,|u_x|^2)\,\mathrm{d}x \leqslant 3\|u\|_1\int_{-\infty}^{\infty}u_x^2\,\mathrm{d}x \leqslant 3\|u\|_1^3, \tag{4.13}$$

and the latter quantity is bounded in terms of only the H^1 norm of g, by proposition 2. Letting C denote a constant, depending monotonically as it happens on the H^3 norm of g, which bounds the first two terms on the right hand side of (4.11) and estimating further the remaining integral on the right of (4.11) leads to

$$\int_{-\infty}^{\infty}u_{xx}^2\,\mathrm{d}x \leqslant C+\epsilon\int_0^t(3\|u_t\|_\infty\|u_{xx}\|^2+3\|u\|_\infty^2\|u_x\|\,\|u_{xt}\|+6\|u_x\|_\infty\|u_{xx}\|\,\|u_{xt}\|)\,\mathrm{d}\tau, \tag{4.14}$$

where

$$\|f\|_\infty = \sup_{x\in\mathbb{R}}|f(x)|. \tag{4.15}$$

Since $\|u\|_1$ and $\|u\|_\infty$ are known from proposition 2 and its corollary to be bounded in terms solely of $\|g\|_1$ independently of ϵ and $t \geqslant 0$, (4.14) implies the following inequality.

$$\|u\|_2^2 \leqslant C+\epsilon C\int_0^t(\|u_t\|_\infty\|u\|_2^2+\|u_{xt}\|+\|u_x\|_\infty\|u\|_2\|u_{xt}\|)\,\mathrm{d}\tau.$$

Here C denotes constants independent of $\epsilon \leqslant \epsilon_1$ and $t \geqslant 0$, but dependent on $\|g\|_3$. Writing for short

$$A = A(t) = \|u\|_2, \tag{4.16}$$

the last inequality comes to

$$A^2(t) \leqslant C+\epsilon C\int_0^t(\|u_t\|_\infty A^2(\tau)+\|u_{xt}\|+\|u_x\|_\infty\|u_{xt}\|A(\tau))\,\mathrm{d}\tau. \tag{4.17}$$

In order to make use of the last inequality, some estimates of time derivatives of u must be derived. To this end, differentiate the regularized equation with respect to t. Letting $v = u_t$, this is written as

$$v_t+(uv)_x+v_{xxx}-\epsilon v_{xxt} = 0.$$

Multiply this equation by v and integrate over $\mathbb{R}$. After a few integrations by parts, there emerges

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(v^2+\epsilon v_x^2)\,\mathrm{d}x = -\int_{-\infty}^{\infty}u_x v^2\,\mathrm{d}x.$$

Now integrate the last relation over $[0, t]$ to obtain

$$\int_{-\infty}^{\infty}(u_t^2+\epsilon u_{xt}^2)\,\mathrm{d}x = \int_{-\infty}^{\infty}[u_t(x,0)^2+\epsilon u_{xt}(x,0)^2]\,\mathrm{d}x-\int_0^t\int_{-\infty}^{\infty}u_x u_t^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

Define

$$B^2 = B^2(t) = \int_{-\infty}^{\infty}(u_t^2+\epsilon u_{xt}^2)\,\mathrm{d}x. \tag{4.18}$$

Then the last relation gives the integral inequality

$$B^2(t) \leqslant B^2(0)+\int_0^t\|u_x\|_\infty B^2(\tau)\,\mathrm{d}\tau. \tag{4.19}$$

The estimates (4.17) and (4.19) are extended by the following elementary inequalities.

LEMMA 3. In the above notation, the following inequalities are valid:

$$\text{(i)} \quad \|u_t\|_\infty \leqslant \epsilon^{-\frac{1}{4}} B(t),$$

$$\text{(ii)} \quad \|u_x\|_\infty \leqslant (\|u_x\|\,\|u_{xx}\|)^{\frac{1}{2}} \leqslant CA(t)^{\frac{1}{2}},$$

$$\text{(iii)} \quad \|u_{xt}\| \leqslant \epsilon^{-\frac{1}{2}} B(t),$$

where C depends on $\|g\|_1$.

Proof (*of Lemma*). (iii) is trivial and (ii) follows from the first half of (4.5) applied to u_x and the bound on $\|u_x\|$ implied by (4.1). For (i) use (4.5) and the elementary inequality $ab \leqslant a^2+b^2$.

$$\begin{aligned}\|u_t\|_\infty^2 &\leqslant \|u_t\|\,\|u_{xt}\| = \epsilon^{-\frac{1}{2}}[\|u_t\|\,(\epsilon^{\frac{1}{2}}\|u_{xt}\|)] \\ &\leqslant \epsilon^{-\frac{1}{2}}[\|u_t\|^2+\epsilon\|u_{xt}\|^2] = \epsilon^{-\frac{1}{2}}B(t)^2.\end{aligned}$$

Taking the square root gives (i) and concludes the proof of the lemma.

From the results of lemma 3, (4.17) and (4.19) yield the following coupled system of integral inequalities.

$$\left.\begin{aligned}A^2(t) &\leqslant C+\epsilon C\int_0^t [\epsilon^{-\frac{1}{4}}BA^2+\epsilon^{-\frac{1}{2}}B+\epsilon^{-\frac{1}{2}}BA^{\frac{3}{2}}]\,\mathrm{d}\tau, \\ B^2(t) &\leqslant B^2(0)+C\int_0^t A^{\frac{1}{2}}B^2\,\mathrm{d}\tau.\end{aligned}\right\} \tag{4.20}$$

The next task is to obtain a bound on $B(0)$ which depends only on the initial data. This calculation is the subject of the next lemma.

LEMMA 4. Let $g \in H^\infty$ and let u be the corresponding solution of the initial-value problem (3.1) for the regularized K.–dV. equation. Then

$$B(0) \leqslant \|g\|_3(\|g\|_1+1). \tag{4.21}$$

Proof. Multiply the regularized equation by u_t and perform one integration by parts to derive the identity

$$\begin{aligned}B^2(t) &= -\int_{-\infty}^{\infty} u_t(uu_x+u_{xxx})\,\mathrm{d}x \\ &\leqslant B(t)\,\|u\|_3(\|u\|_1+1).\end{aligned} \tag{4.22}$$

Cancelling $B(t)$ in (4.22) gives an estimate for $B(t)$:

$$B(t) \leqslant \|u\|_3(\|u\|_1+1),$$

which at $t=0$ yields the required bound (4.21).

Thus again letting C denote constants dependent only on $\|g\|_k$ for $k \leqslant 3$, and not on T or ϵ in $(0,\epsilon_1]$, the system (4.20) becomes the following coupled system of integral inequalities.

$$A^2 \leqslant C+\epsilon C\int_0^t (\epsilon^{-\frac{1}{4}}BA^2+\epsilon^{-\frac{1}{2}}B+\epsilon^{-\frac{1}{2}}BA^{\frac{3}{2}})\,\mathrm{d}\tau,$$

$$B^2 \leqslant C+C\int_0^t A^{\frac{1}{2}}B^2\,\mathrm{d}\tau.$$

Upon defining the new dependent variable $D^2(t) = A^2(t)+1$, the above formulae imply

$$\left.\begin{aligned} D^2 &\leqslant C+\epsilon^{\frac{1}{2}}C\int_0^t BD^2\,\mathrm{d}\tau, \\ B^2 &\leqslant C+C\int_0^t D^{\frac{1}{2}}B^2\,\mathrm{d}\tau. \end{aligned}\right\} \tag{4.23}$$

There is a particularly convenient form in which to write the constants in (4.23). The claim is that (4.23) implies

$$\left.\begin{aligned} D^2 &\leqslant \left(\frac{\alpha}{1-\epsilon^{\frac{1}{2}}}\right)^4+\epsilon^{\frac{1}{2}}\frac{4\gamma}{\beta}\int_0^t D^2B\,\mathrm{d}\tau, \\ B^2 &\leqslant \left(\frac{\beta}{1-\epsilon^{\frac{1}{2}}}\right)^2+\frac{2\gamma}{\alpha}\int_0^t D^{\frac{1}{2}}B^2\,\mathrm{d}\tau, \end{aligned}\right\} \tag{4.24}$$

where α, β, γ do not depend on $\epsilon \leqslant \epsilon_1$, provided $\epsilon_1 < 1$ which is henceforth assumed. (First choose $\epsilon_1 < 1$ in accordance with the previous restrictions (4.10), then choose α, β sufficiently large, and finally choose γ sufficiently large. Since the constants C in (4.23) can all be assumed to depend monotonically on $\|g\|_3$, then without loss of generality, α, β and γ do as well, though this is not crucial in what follows.) Now define $\bar{D}$ and $\bar{B}$ by

$$\left.\begin{aligned} \bar{D}^2 &= \left(\frac{\alpha}{1-\epsilon^{\frac{1}{2}}}\right)^4+\epsilon^{\frac{1}{2}}\frac{4\gamma}{\beta}\int_0^t \bar{D}^2\bar{B}\,\mathrm{d}\tau, \\ \bar{B}^2 &= \left(\frac{\beta}{1-\epsilon^{\frac{1}{2}}}\right)^2+\frac{2\gamma}{\alpha}\int_0^t \bar{D}^{\frac{1}{2}}\bar{B}^2\,\mathrm{d}\tau. \end{aligned}\right\} \tag{4.25}$$

Then by their construction it is always the case that $D \leqslant \bar{D}$ and $B \leqslant \bar{B}$, for all $t \geqslant 0$. But $\bar{D}$ and $\bar{B}$ can be determined explicitly as

$$\bar{D} = \left(\frac{\alpha}{1-\epsilon^{\frac{1}{2}}\,\mathrm{e}^{\gamma t}}\right)^2, \quad \bar{B} = \left(\frac{\beta\,\mathrm{e}^{\gamma t}}{1-\epsilon^{\frac{1}{2}}\,\mathrm{e}^{\gamma t}}\right). \tag{4.26}$$

If ϵ_2 is chosen so that $1-\epsilon_2^{\frac{1}{2}}\mathrm{e}^{\gamma T} \geqslant \frac{1}{2}$ say, and $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$, then since γ depends only on $\|g\|_3$, $\epsilon_0 = \epsilon_0(T, \|g\|_3)$. Further, if $0 < \epsilon \leqslant \epsilon_0$, (4.26) then shows explicitly that D and B are bounded on $[0, T]$, independent of $\epsilon \leqslant \epsilon_0$, with a bound depending only on T and the H^3 norm of the initial data. Thus $\|u_t\|$ and $\|u\|_2$ are seen to be bounded on $[0, T]$, independently of ϵ in $(0, \epsilon_0]$. Note in particular from the explicit form of $\bar{D}$ that because the constant α depends (monotonically) on $\|g\|_3$, at least for $\epsilon \leqslant \epsilon_1$, we can now write $\|u\|_2 \leqslant a_1(\|g\|_3)$ on $[0, T]$ for ϵ in $(0, \epsilon_0]$ where a_1 is continuous, monotone increasing and $a_1(0) = 0$. It is worth note that for *all* $t \geqslant 0$,

$$\limsup_{\epsilon \downarrow 0} \|u_{xx}\| \leqslant \alpha = \alpha(\|g\|_3). \tag{4.27}$$

The final stage of the derivation of *a priori* bounds is the bounding of derivatives of order higher than 2.

Proposition 4. Let $T > 0$ and $g \in H^\infty$ be given. Let ϵ_0 be as in proposition 3. Then for $\epsilon \leqslant \epsilon_0$, the solution u to the regularized initial-value problem (3.1) is bounded in $\mathscr{H}_T^m$, for all $m \geqslant 3$, with a bound depending only on T, ϵ_0, $\|g\|_m$ and $\epsilon^{\frac{1}{2}}\|g\|_{m+1}$.

Proof. The argument is made by induction. It is known from the last proposition that u is bounded in $\mathscr{H}_T^2$ with a bound depending only on T, ϵ_0 and $\|g\|_3$. Let $m > 2$ and suppose inductively that u is bounded in $\mathscr{H}_T^{m-1}$ independently of ϵ in $(0, \epsilon_0]$ with a bound depending only on

T, ϵ_0 and $\|g\|_m$. It will be shown that u is then bounded in $\mathscr{H}_T^m$ with a bound depending only on T, ϵ_0, $\|g\|_m$ and $\epsilon^{\frac{1}{2}}\|g\|_{m+1}$. (This allows the induction to commence with the already obtained $\mathscr{H}_T^2$ bounds and yet still gives the desired bounds on u in $\mathscr{H}_T^m$ in terms of $\|g\|_m$ and $\epsilon^{\frac{1}{2}}\|g\|_{m+1}$.) Introduce the notation

$$u_{(k)} = \partial_x^k u$$

for spatial differentiation. Multiply the regularized equation by $u_{(2m)}$ and integrate over $\mathbb{R}$. Integrating by parts appropriately yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}[u_{(m)}^2+\epsilon u_{(m+1)}^2]\,\mathrm{d}x = -\int_{-\infty}^{\infty}(u^2)_{(m+1)}u_{(m)}\,\mathrm{d}x. \tag{4.28}$$

Now use Leibnitz' rule to expand the first term in the integrand on the right hand side. Since u is known to be bounded in $\mathscr{H}_T^{m-1}$, independently of ϵ in $(0, \epsilon_0]$, it follows that

$$\left.\begin{aligned}\|u_{(r)}\| &\leqslant C \quad \text{for} \quad r \leqslant m-1,\\ \|u_{(r)}\|_\infty &\leqslant C \quad \text{for} \quad r \leqslant m-2,\end{aligned}\right\} \tag{4.29}$$

where $C = C(T, \epsilon_0, \|g\|_m)$.

The right hand integral in (4.28) can be estimated as follows.

$$\int_{-\infty}^{\infty}(u^2)_{(m+1)}u_{(m)}\,\mathrm{d}x = \int_{-\infty}^{\infty}\left[c_0 u u_{(m+1)}u_{(m)} + c_1 u_x u_{(m)}^2 + u_{(m)}\sum_{r=2}^{m-2} c_r u_{(r)}u_{(m+1-r)} + u_{(m-1)}^2 u_{(m)}\right]\mathrm{d}x. \tag{4.30}$$

The last term in the right hand integrand of (4.30) only occurs in the case $m = 3$, and since it is a perfect x-derivative, it integrates to 0 in any case. Hence it is ignored in the following. The first term on the right of (4.30) is conveniently integrated by parts,

$$\int_{-\infty}^{\infty} u u_{(m+1)}u_{(m)}\,\mathrm{d}x = -\frac{1}{2}\int_{-\infty}^{\infty} u_x u_{(m)}^2\,\mathrm{d}x,$$

and then combined with the second term in the same integral. Since $m \geqslant 3$, $\|u_x\|_\infty \leqslant C$ from (4.29). There follows the estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}[u_{(m)}^2+\epsilon u_{(m+1)}^2]\,\mathrm{d}x &\leqslant C\int_{-\infty}^{\infty}u_{(m)}^2\,\mathrm{d}x + \sum_{r=2}^{m-2} c_r\|u_{(m)}\|\,\|u_{(r)}\|_\infty\,\|u_{(m+1-r)}\|\\ &\leqslant C\int_{-\infty}^{\infty}u_{(m)}^2\,\mathrm{d}x + C\left[\int_{-\infty}^{\infty}u_{(m)}^2\,\mathrm{d}x\right]^{\frac{1}{2}} \leqslant C\left(\int_{-\infty}^{\infty}u_{(m)}^2\,\mathrm{d}x+1\right).\end{aligned}$$

Define
$$E_m(t) = \int_{-\infty}^{\infty}[u_{(m)}^2+\epsilon u_{(m+1)}^2]\,\mathrm{d}x.$$

Then the last inequality implies

$$\mathrm{d}E_m/\mathrm{d}t \leqslant C(E_m+1),$$

from which there follows immediately that for $t \geqslant 0$,

$$E_m(t) \leqslant E_m(0)\,\mathrm{e}^{Ct}+\mathrm{e}^{Ct}-1,$$

independently of $\epsilon \leqslant \epsilon_0$ and t in $[0, T]$. The bound for $E_m(t)$ is seen explicitly to depend only on $\|g\|_m$ and $\epsilon^{\frac{1}{2}}\|g\|_{m+1}$ since C depends only on $\|g\|_m$ and

$$E_m(0)^{\frac{1}{2}} \leqslant \|g\|_m + \epsilon^{\frac{1}{2}}\|g\|_{m+1}.$$

Thus $\|u_{(m)}\| \leqslant E_m^{\frac{1}{2}}$ is bounded on $[0, T]$, and hence u is bounded in $\mathscr{H}_T^m$, independently of ϵ in $(0, \epsilon_0]$, and the proposition is proved.

Corollary. The solution u to the regularized initial-value problem (3.1) corresponding to a given $g \in H^\infty$ is bounded in $\mathscr{H}_T^{k,l}$, independently of $\epsilon \leqslant \epsilon_0$, for all k, l, and $T \geqslant 0$.

Proof. This follows from the last proposition by judicious use of the regularized differential equation. Specifically, write it in the form

$$(1-\epsilon\,\partial_x^2)\,u_t = -uu_x - u_{xxx}.$$

Invert the operator $1-\epsilon\,\partial_x^2$, as in lemma 1, to come to the expression

$$u_t = -K_\epsilon * (uu_x + u_{xxx}),$$

where $\hat{K}_\epsilon(k) = (1+\epsilon k^2)^{-1}$. A simple estimate in Fourier transformed variables shows that if $V \in \mathscr{H}_T^m$, then $K_\epsilon * V$ is bounded in $\mathscr{H}_T^m$ independently of $\epsilon \geqslant 0$. Therefore since for each $m \geqslant 0$ the right hand side of the last display is bounded in $\mathscr{H}_T^m$, independently of ϵ in $(0, \epsilon_0]$, so u_t is bounded in $\mathscr{H}_T^m$ for each $m \geqslant 0$, independently of small enough ϵ. This shows that $\partial_x^k u_t$ is bounded in $\mathscr{H}_T$ for each $k \geqslant 0$, that is u is bounded in $\mathscr{H}_T^{m,1}$ for each m, independently of ϵ in $(0, \epsilon_0]$. A straightforward induction argument now finishes the proof.

5. Convergence of the approximations

In this section the behaviour of solutions of the regularized K.–dV. equation as $\epsilon \downarrow 0$ is considered. It is shown that strong convergence to solutions of the pure initial-value problem for the K.–dV. equation obtains. The method of proof is elementary. Given initial data $g \in H^k$, where $k \geqslant 3$, first regularize g by convolution with a smooth function (an approximate identity). Then pose and solve the regularized K.–dV. equation for this smoothed data. Finally it appears that the solutions obtained from this process form a Cauchy sequence in the appropriate function space, and hence converge strongly. The limiting function is then a solution to the initial-value problem for the K.–dV. equation. The starting point is a few preliminary lemmas which will set the stage for the main result.

Let $g \in H^s$ where $s \geqslant 3$ and let $\epsilon > 0$. Define a regularization g_ϵ of g as follows (here, as before, $\hat{f}$ denotes the Fourier transform of f)

$$\hat{g}_\epsilon(k) = \varphi(\epsilon^{\frac{1}{6}}k)\,\hat{g}(k), \tag{5.1}$$

where φ is an even C^∞ function, with $0 \leqslant \varphi \leqslant 1$ everywhere and $\varphi(0) = 1$, such that $\psi(k) = 1-\varphi(k)$ has a zero of infinite order at 0 and such that φ tends exponentially to 0 at $\pm\infty$. Such functions are abundant. For example, $\varphi(k) = \mathrm{e}^{-g(k)}$ where $g(k) = k^2\,\mathrm{e}^{-1/k^2}$ has the desired properties. It is immediate that $g_\epsilon \in H^\infty$. Hence there is a unique C^∞ solution $u_\epsilon(x,t) = u(x,t,\epsilon)$, all of whose derivatives lie in $\mathscr{H}_T$ for all $T > 0$, to the regularized K.–dV. equation with regularized data g_ϵ:

$$\left.\begin{aligned} u_t + uu_x + u_{xxx} - \epsilon u_{xxt} &= 0,\\ u(x,0) &= g_\epsilon(x). \end{aligned}\right\} \tag{5.2}$$

Here is a simple result giving bounds on various norms of g_ϵ in terms of norms of g.

Lemma 5. Let $g \in H^s$ where $s \geqslant 3$ and let g_ϵ be the smoothed version of g defined in (5.1). Then as $\epsilon \downarrow 0$,

$$\left.\begin{aligned} \|g_\epsilon\|_{s+j} &= O(\epsilon^{-\frac{1}{6}j}) \quad \text{for} \quad j = 1, 2, \ldots\\ \|g - g_\epsilon\|_{s-j} &= o(\epsilon^{\frac{1}{6}j}) \quad \text{for} \quad j = 1, 2, \ldots\\ \|g - g_\epsilon\|_s &= o(1). \end{aligned}\right\} \tag{5.3}$$

Furthermore, the first bound holds uniformly on bounded subsets of H^s, and the last two bounds hold uniformly on compact subsets of H^s. The second bound holds uniformly on bounded subsets of H^s if $o(\epsilon^{\frac{1}{6}j})$ is replaced by $O(\epsilon^{\frac{1}{6}j})$.

Proof. This is an easy calculation in the Fourier transformed variables.

$$\begin{aligned}\epsilon^{\frac{1}{3}j}\|g_\epsilon\|_{s+j}^2 &= \epsilon^{\frac{1}{3}j}\int_{-\infty}^{\infty}[1+k^2+\ldots+k^{2(s+j)}]\,|\hat{g}_\epsilon(k)|^2\,\mathrm{d}k \\ &= \int_{-\infty}^{\infty}\left[\epsilon^{\frac{1}{3}j}\frac{1+\ldots+k^{2(s+j)}}{1+\ldots+k^{2s}}\varphi^2(\epsilon^{\frac{1}{6}}k)\right][1+\ldots+k^{2s}]\,|\hat{g}(k)|^2\,\mathrm{d}k \\ &\leqslant \sup_{k\in\mathbb{R}}\left[\epsilon^{\frac{1}{3}j}\frac{1+\ldots+k^{2(s+j)}}{1+\ldots+k^{2s}}\varphi^2(\epsilon^{\frac{1}{6}}k)\right]\|g\|_s^2.\end{aligned}$$

Letting $K=\epsilon^{\frac{1}{6}}k$ and $\gamma=\epsilon^{\frac{1}{3}}$, the last inequality can be continued more transparently. Since $0<\epsilon\leqslant 1$, so $0<\gamma\leqslant 1$. Hence

$$\begin{aligned}\epsilon^{\frac{1}{3}j}\|g_\epsilon\|_{s+j}^2 &\leqslant \|g\|_s^2\sup_{K\in\mathbb{R}}\gamma^j\frac{1+\ldots+(K^2/\gamma)^{s+j}}{1+\ldots+(K^2/\gamma)^s}\varphi^2(K) \\ &\leqslant \|g\|_s^2\sup_{K\in\mathbb{R}}\left[\frac{\gamma^{s+j}+\ldots+K^{2(s+j)}}{\gamma^s+\ldots+K^{2s}}\right]\varphi^2(K).\end{aligned}$$

Estimating separately the ranges $|K|<1$ and $|K|\geqslant 1$ leads to the bound

$$\epsilon^{\frac{1}{3}j}\|g_\epsilon\|_{s+j}^2 \leqslant \|g\|_s^2(s+j)\sup_{K\in\mathbb{R}}\{1+K^{2j}\varphi^2(K)\}.$$

That is,

$$\epsilon^{\frac{1}{6}j}\|g_\epsilon\|_{s+j}\leqslant C\|g\|_s,$$

where C is independent of g and of ϵ. This establishes that the first bound in (5.3) holds uniformly for bounded subsets of H^s. The third conclusion must be handled a little differently. First note that if $g\in H^s$, then

$$\|g-g_\epsilon\|_s^2=\int_{-\infty}^{\infty}\psi^2(\epsilon^{\frac{1}{6}}k)\,[1+\ldots+k^{2s}]\,|\hat{g}(k)|^2\,\mathrm{d}k. \tag{5.4}$$

As $\epsilon\downarrow 0$, the integrand tends pointwise to zero almost everywhere. Further, the integrand is bounded above by the integrable function

$$(1+k^2+\ldots+k^{2s})\,|\hat{g}(k)|^2.$$

Hence Lebesgue's dominated convergence theorem applies and we may conclude $\|g-g_\epsilon\|_s\to 0$ as $\epsilon\downarrow 0$.

Second, note that to demonstrate the uniformity on compact subsets, it is enough to show that

$$g_n\to g \text{ in } H^s \Rightarrow \|g_{n\epsilon}-g_n\|_s\to 0 \text{ as } \epsilon\downarrow 0 \quad \text{uniformly for}\quad n=1,2,\ldots, \tag{5.5}$$

since sequential compactness is equivalent to compactness in a metric space.

To prove this, let $\gamma>0$ be given. It is required to find an $\epsilon_0>0$ so that if $0<\epsilon\leqslant\epsilon_0$, then $\|g_{n\epsilon}-g_n\|_s<\gamma$ for all $n=1,2,\ldots$. Notice that for all n,

$$\begin{aligned}\|g_{n\epsilon}-g_\epsilon\|_s^2 &= \|(g_n-g)_\epsilon\|^2 \\ &= \int_{-\infty}^{\infty}\psi^2(\epsilon^{\frac{1}{6}}k)\,(1+\ldots+k^{2s})\,|\hat{g}_n(k)-\hat{g}(k)|^2\,\mathrm{d}k \\ &\leqslant \int_{-\infty}^{\infty}(1+\ldots+k^{2s})\,|\hat{g}_n(k)-\hat{g}(k)|^2\,\mathrm{d}k = \|g_n-g\|_s^2.\end{aligned} \tag{5.6}$$

Hence to verify (5.5), choose N so large that if $n \geqslant N$, then $\|g_n - g\|_s < \frac{1}{3}\gamma$. Then choose ϵ_0 so small that $\|g_{k\epsilon} - g_k\|_s < \frac{1}{3}\gamma$ for $1 \leqslant k \leqslant N$ and $\|g_\epsilon - g\|_s < \frac{1}{3}\gamma$, for ϵ in $(0, \epsilon_0]$. Then certainly

$$\|g_{n\epsilon} - g_n\|_s < \gamma, \tag{5.7}$$

for $1 \leqslant n \leqslant N$. If $n \geqslant N$, then by using (5.6),

$$\begin{aligned}\|g_{n\epsilon} - g_n\|_s &\leqslant \|g_{n\epsilon} - g_\epsilon\|_s + \|g_\epsilon - g\|_s + \|g - g_n\|_s \\ &\leqslant \|g_n - g\|_s + \|g_\epsilon - g\|_s + \|g - g_n\|_s \\ &\leqslant \tfrac{1}{3}\gamma + \tfrac{1}{3}\gamma + \tfrac{1}{3}\gamma = \gamma.\end{aligned}$$

Hence (5.7) holds for all n.

For the second inequality in (5.3) a similar argument applies,

$$\begin{aligned}\|g - g_\epsilon\|^2_{s-j} &= \int_{-\infty}^{\infty} \left[\frac{1 + \ldots + k^{2(s-j)}}{1 + \ldots + k^{2s}} \psi^2(\epsilon^{\frac{1}{6}}k)\right] [1 + \ldots + k^{2s}]\, |\hat{g}(k)|^2\, \mathrm{d}k \\ &\leqslant \sup_{k \in \mathbb{R}} \left[\frac{1 + \ldots + k^{2(s-j)}}{1 + \ldots + k^{2s}} \psi(\epsilon^{\frac{1}{6}}k)\right] \int_{-\infty}^{\infty} \psi(\epsilon^{\frac{1}{6}}k)\, [1 + \ldots + k^{2s}]\, |\hat{g}(k)|^2\, \mathrm{d}k \\ &\leqslant C\epsilon^{\frac{1}{3}j} \int_{-\infty}^{\infty} \psi(\epsilon^{\frac{1}{6}}k)\, [1 + \ldots + k^{2s}]\, |\hat{g}(k)|^2\, \mathrm{d}k,\end{aligned}$$

where again C denotes a constant which does not depend on g or on $\epsilon \leqslant 1$. As in the proof just given above that $\|g - g_\epsilon\|_s = o(1)$, the integral on the right side of the last display is $o(1)$ as $\epsilon \downarrow 0$, uniformly on compact subsets of H^s. The integral is also bounded above by $\|g\|_s^2$ so that $\|g - g_\epsilon\|_{s-j} = O(\epsilon^{\frac{1}{6}j})$ uniformly on bounded subsets of H^s as well. Thus all three parts of (5.3) are verified and the lemma is established.

Corollary 1. Let $s \geqslant 3$. Then u_ϵ is bounded in $\mathscr{H}^s_T$ independently of sufficiently small ϵ for each finite $T > 0$. Further, $\epsilon^{\frac{1}{6}m} u_\epsilon$ is bounded in $\mathscr{H}^{s+m}_T$ independently of sufficiently small ϵ for each finite $T > 0$ and $m \geqslant 1$.

Proof. This follows from the last lemma and the results of proposition 4. For by proposition 4, $\|u_\epsilon\|_s$ has an upper bound depending only on T, ϵ_0, $\|g_\epsilon\|_s$ and $\epsilon^{\frac{1}{2}}\|g_\epsilon\|_{s+1}$. From the properties of the regularization $\|g_\epsilon\|_s \leqslant \|g\|_s$ and $\epsilon^{\frac{1}{2}}\|g_\epsilon\|_{s+1} \leqslant C\epsilon^{\frac{1}{3}}\|g\|_s$. Hence, independently of t in $[0, T]$ and of sufficiently small ϵ, $\|u_\epsilon\|_s$ has an upper bound depending only on T, ϵ_0 and $\|g\|_s$. A similar proof yields the $\mathscr{H}^{s+m}_T$ bounds on $\epsilon^{\frac{1}{6}m}u_\epsilon$.

Corollary 2. $\partial_t u_\epsilon$ is bounded in $\mathscr{H}^{s-3}_T$ and $\epsilon^{\frac{1}{6}m}\partial_x^{s+m-3}\partial_t u_\epsilon$ is bounded in $\mathscr{H}_T$ independently of sufficiently small ϵ, for all finite $T > 0$ and $m = 1, 2, \ldots, 5$.

Proof. Use the method, previously elucidated in the proof of lemma 1, of inverting $(1 - \epsilon\,\partial_x^2)$ in the regularized equation (5.2) to gain the estimate

$$\begin{aligned}\|\partial_t u_\epsilon\|_{s-3} &\leqslant \|u_\epsilon\|_{s-3}\|\partial_x u_\epsilon\|_{s-3} + \|\partial_x^3 u_\epsilon\|_{s-3} \\ &\leqslant \|u_\epsilon\|_s^2 + \|u_\epsilon\|_s,\end{aligned} \tag{5.8}$$

and similarly

$$\begin{aligned}\epsilon^{\frac{1}{6}m}\|\partial_t u_\epsilon\|_{s+m-3} &\leqslant \epsilon^{\frac{1}{6}m}(\|u_\epsilon\|_{s+m-3}\|u_\epsilon\|_{s+m-2} + \|u_\epsilon\|_{s+m}) \\ &\leqslant C,\end{aligned} \tag{5.9}$$

where C is a constant, independent of ϵ sufficiently small by corollary 1 since $m \leqslant 5$.

The major proposition of this section states that the functions $\{u_\epsilon\}$ are Cauchy in $\mathscr{H}^s_T$ as $\epsilon \downarrow 0$.

PROPOSITION. 5. Let u_ϵ be the solution of the regularized K.-dV. equation corresponding to the initial data g_ϵ, as in (5.2), where $g \in H^s$ and $s \geqslant 3$. Then $\{u_\epsilon\}$ is Cauchy in $\mathscr{H}^s_T$ as $\epsilon \downarrow 0$.

Proof. Let $u = u_\epsilon$ and $v = u_\delta$, where $\delta \leqslant \epsilon$ say. It suffices to show that $\|u-v\|_s$ can be made as small as we like, for t in $[0, T]$, by taking ϵ sufficiently small. To this end, define $w = u-v$. Then w satisfies the partial differential equation

$$w_t + (uw + \tfrac{1}{2}w^2)_x + w_{xxx} - \delta w_{xxt} = (\epsilon-\delta)\, u_{xxt}, \tag{5.10}$$

with $w(x, 0) = g_\epsilon(x) - g_\delta(x) = h(x)$, say. For $j \leqslant s$, the identities

$$\int_{-\infty}^{\infty} [w_{(j)}^2 + \delta w_{(j+1)}^2]\, dx = \int_{-\infty}^{\infty} [h_{(j)}^2 + \delta h_{(j+1)}^2]\, dx - 2\int_0^t \int_{-\infty}^{\infty} [(uw+\tfrac{1}{2}w^2)_{(j+1)} - (\epsilon-\delta)\, u_{t,(j+2)}]\, w_{(j)}\, dx\, d\tau, \tag{5.11}$$

are derived by multiplying (5.10) by $w_{(2j)}$, integrating over $\mathbb{R}$ and over $[0, t]$ and by using partial integration. (Here as before $w_{(j)}$ is shorthand notation for $\partial_x^j w$.) We work out the details for the case $s = 3$, and then show how $s > 3$ goes by induction.

Consider the case $j = 0$ in (5.11). Let

$$V_0(t)^2 = \int_{-\infty}^{\infty} [w^2 + \delta w_x^2]\, dx,$$

so that

$$V_0(0)^2 = \int_{-\infty}^{\infty} [h^2 + \delta h_x^2]\, dx.$$

Then (5.11) can be put in the form

$$V_0(t)^2 = V_0(0)^2 - 2\int_0^t \int_{-\infty}^{\infty} [(w_x + \tfrac{1}{2}u_x)\, w^2]\, dx\, d\tau + 2(\epsilon-\delta)\int_0^t \int_{-\infty}^{\infty} u_{xxt}\, w\, dx\, d\tau.$$

Now it follows from corollary 1 above that $|w_x + \frac{1}{2}u_x|$ is bounded on $[0, T]$, say by a constant C_1 depending on T and $\|g\|_3$, independent of sufficiently small ϵ. Further from corollary 2 above, $\epsilon^{\frac{1}{3}}\|u_{xxt}\|$ is bounded on $[0, T]$, say by C_2 where C_2 depends on T and on $\|g\|_3$, but is independent of sufficiently small ϵ. Hence for sufficiently small ϵ,

$$V_0(t)^2 \leqslant V_0(0)^2 + 2C_1 \int_0^t V_0(\tau)^2\, d\tau + 2\epsilon^{\frac{2}{3}} C_2 \int_0^t V_0(\tau)\, d\tau.$$

It follows readily that for t in $[0, T]$,

$$\|w\| \leqslant V_0(t) \leqslant V_0(0)\, e^{C_1 T} + \epsilon^{\frac{2}{3}} C_2 (e^{C_1 T} - 1)\, C_1^{-1}.$$

Now,

$$V_0(0) = \left\{\int_{-\infty}^{\infty} [(g_\delta(x) - g_\epsilon(x))^2 + \delta(g_\delta'(x) - g_\epsilon'(x))^2]\, dx\right\}^{\frac{1}{2}}$$
$$\leqslant \|g_\delta - g\|_1 + \|g_\epsilon - g\|_1 \leqslant C\epsilon^{\frac{1}{3}},$$

as $\epsilon \downarrow 0$ by the lemma 5 above. Hence $\{u_\epsilon\}$ is seen to be Cauchy in $\mathscr{H}_T$ from the estimate $\|w\| \leqslant C\epsilon^{\frac{1}{3}}$, valid for ϵ sufficiently small.

Consider next $j = 1$. Again, for convenience of writing, define

$$V_1(t)^2 = \int_{-\infty}^{\infty} [w_x^2 + \delta w_{xx}^2]\, dx,$$

so that

$$V_1(0)^2 = \int_{-\infty}^{\infty} [h'^2 + \delta h''^2]\, dx.$$

Now integrate (5.11) by parts to come to

$$V_1(t)^2 = V_1(0)^2 - 2\int_0^t\int_{-\infty}^{\infty} (\tfrac{1}{2}w_x + \tfrac{3}{2}u_x)\, w_x^2 \,\mathrm{d}x\,\mathrm{d}\tau - 2\int_0^t\int_{-\infty}^{\infty} [u_{xx} w - (\epsilon-\delta)\, u_{xxxt}]\, w_x \,\mathrm{d}x\,\mathrm{d}\tau.$$

But by corollary 1 and 2 above, on $[0, T]$, $|\tfrac{1}{2}w_x + \tfrac{3}{2}u_x|$, $|u_{xx}|$, and $\epsilon^{\frac{1}{2}}\|u_{xxxt}\|$ are all bounded independently of sufficiently small ϵ. Also, as mentioned above, $\|w\| \leqslant C\epsilon^{\frac{1}{3}}$ on $[0, T]$ for ϵ small enough. Hence for ϵ sufficiently small,

$$\begin{aligned} V_1(t)^2 &\leqslant V_1(0)^2 + 2C\int_0^t (\|w_x\|^2 + \epsilon^{\frac{1}{3}}\|w_x\|)\,\mathrm{d}\tau \\ &\leqslant V_1(0)^2 + 2C\int_0^t (V_1(\tau)^2 + \epsilon^{\frac{1}{3}} V_1(\tau))\,\mathrm{d}\tau, \end{aligned}$$

where here, and in the remainder of the proof, C denotes various constants depending on T and on norms of g up to order s, but independent of sufficiently small ϵ. From this it follows instantly that on $[0, T]$,

$$V_1(t) \leqslant V_1(0)\,\mathrm{e}^{Ct} + \epsilon^{\frac{1}{3}}(\mathrm{e}^{Ct}-1).$$

Hence, for t in $[0, T]$

$$\|w_x\| \leqslant V_1(t) \leqslant V_1(0)\,\mathrm{e}^{CT} + \epsilon^{\frac{1}{3}}(\mathrm{e}^{CT}-1).$$

Again,

$$\begin{aligned} V_1(0) &\leqslant \|g-g_\epsilon\|_1 + \|g-g_\delta\|_1 + \delta^{\frac{1}{2}}\|g-g_\epsilon\|_2 + \delta^{\frac{1}{2}}\|g-g_\delta\|_2 \\ &\leqslant C\epsilon^{\frac{1}{3}} \end{aligned}$$

as $\epsilon \downarrow 0$, ($\delta < \epsilon$ in these calculations) by lemma 5. It is now apparent that $\{u_\epsilon\}$ is Cauchy in $\mathscr{H}^1_T$, and that in fact, on $[0, T]$, for ϵ sufficiently small,

$$\|w\|_1 \leqslant C\epsilon^{\frac{1}{3}}. \tag{5.12}$$

For $j = 2$, (5.11) comes to

$$V_2(t)^2 = V_2(0)^2 - 2\int_0^t\int_{-\infty}^{\infty} [(uw + \tfrac{1}{2}w^2)_{xxx} w_{xx} - (\epsilon-\delta)\, u_{xxxxt} w_{xx}]\,\mathrm{d}x\,\mathrm{d}\tau, \tag{5.13}$$

where

$$V_2(t)^2 = \int_{-\infty}^{\infty} [w_{xx}^2 + \delta w_{xxx}^2]\,\mathrm{d}x.$$

Now since corollary 2 above assures that $\epsilon^{\frac{2}{3}}\|u_{xxxxt}\|$ is bounded for ϵ sufficiently small, the second term in the right hand side in (5.13) is bounded above by $\epsilon^{\frac{1}{3}}C\|w_{xx}\|$. Differentiating and collecting terms in the first summand under the integral leads to

$$\int_0^t\int_{-\infty}^{\infty} (-\tfrac{5}{2}(u_x + w_x)\, w_{xx}^2 - 3u_{xx} w_x w_{xx} - u_{xxx} w w_{xx})\,\mathrm{d}x. \tag{5.14}$$

Now by using (5.12) and corollary 1 to lemma 5 for bounds on u independent of sufficiently small ϵ, we see that on $[0, T]$,

$$\left.\begin{aligned} &\text{(i)} && |u_x + w_x| \leqslant C, \\ &\text{(ii)} && |u_{xx}| \leqslant C, \\ &\text{(iii)} && \|u_{xxx}\| \leqslant C, \\ &\text{(iv)} && \|w_x\| \leqslant C\epsilon^{\frac{1}{3}}, \\ &\text{(v)} && |w| \leqslant C\epsilon^{\frac{1}{3}}, \end{aligned}\right\} \tag{5.15}$$

where C denotes various ϵ-independent constants. It follows from (5.15) applied to (5.14) that the absolute value of the integral in (5.14) is bounded above by

$$C\int_0^t (\|w_{xx}\|^2 + \epsilon^{\frac{1}{3}}\|w_{xx}\|)\,\mathrm{d}\tau.$$

In sum, we have the inequality

$$V_2(t)^2 \leqslant V_2(0)^2 + 2C\int_0^t (\|w_{xx}\|^2 + \epsilon^{\frac{1}{3}}\|w_{xx}\|)\,d\tau$$
$$\leqslant V_2(0)^2 + 2C\int_0^t (V_2(\tau)^2 + \epsilon^{\frac{1}{3}}V_2(\tau))\,d\tau,$$

so that on $[0, T]$ at least, $$\|w_{xx}\| \leqslant V_2(t) \leqslant V_2(0)\,e^{CT} + \epsilon^{\frac{1}{3}}(e^{CT}-1). \tag{5.16}$$

From lemma 5, $$V_2(0) \leqslant \|h\|_2 + \delta^{\frac{1}{2}}\|h\|_3$$
$$\leqslant \|g-g_\epsilon\|_2 + \|g-g_\delta\|_2 + \delta^{\frac{1}{2}}\|g-g_\epsilon\|_3 + \delta^{\frac{1}{2}}\|g-g_\delta\|_3$$
$$\leqslant C'\epsilon^{\frac{1}{6}} + C'\epsilon^{\frac{1}{2}} \leqslant C\epsilon^{\frac{1}{6}}.$$

Putting this in (5.16) yields $$\|w_{xx}\| \leqslant C\epsilon^{\frac{1}{6}}, \tag{5.17}$$

on $[0, T]$. Note that if $s > 3$, we would obtain $C\epsilon^{\frac{1}{3}}$ as bound at this stage, simply because lemma 5 would then allow the bound $V_2(0) \leqslant C\epsilon^{\frac{1}{3}}$.

Finally, consider the highest order case $j = 3$. Define $V_3(t)$ in the, by now, obvious fashion. Then (5.11) gives

$$V_3(t)^2 = V_3(0)^2 - 2\int_0^t\int_{-\infty}^{\infty} [(uw + \tfrac{1}{2}w^2)_{xxxx}w_{xxx} - (\epsilon-\delta)\,u_{xxxxxt}w_{xxx}]\,dx\,d\tau.$$

Since $\epsilon^{\frac{5}{6}}\|u_{xxxxxt}\|$ is bounded, the second term under the integral above converges to 0 as $\epsilon^{\frac{1}{6}}$. Again carrying out the differentiation and integrating by parts, the following expression for the first term under the integral is obtained:

$$2\int_0^t\int_{-\infty}^{\infty} (\tfrac{7}{2}(u_x+w_x)\,w_{xxx}^2 - 4w_xu_{xxx}w_{xxx} - 6u_{xx}w_{xx}w_{xxx} - u_{xxxx}ww_{xxx})\,dx\,d\tau. \tag{5.18}$$

From the corollary to lemma 5 and the results (5.12) and (5.17) already in hand, these estimates hold for t in $[0, T]$

$$\left.\begin{array}{rl}
\text{(i)} & |u_x+w_x| \leqslant C,\\
\text{(ii)} & \|u_{xx}\| \leqslant C,\\
\text{(iii)} & \|u_{xxx}\| \leqslant C,\\
\text{(iv)} & \|u_{xxxx}\| \leqslant C\epsilon^{-\frac{1}{6}},\\
\text{(v)} & |w| \leqslant C\epsilon^{\frac{1}{3}},\\
\text{(vi)} & |w_x| \leqslant C\epsilon^{\frac{1}{6}},\\
\text{(vii)} & \|w_{xx}\| \leqslant C\epsilon^{\frac{1}{6}}.
\end{array}\right\} \tag{5.19}$$

Use (5.19) in (5.18) to derive the upper bound

$$2\int_0^t (C\|w_{xxx}\|^2 + C\epsilon^{\frac{1}{6}}\|w_{xxx}\|)\,d\tau$$

for (5.18). So, for t in $[0, T]$,

$$V_3(t)^2 \leqslant V_3(0)^2 + 2C\int_0^t [V(\tau)^2 + \epsilon^{\frac{1}{6}}V(\tau)]\,d\tau,$$

whence $$\|w_{xxx}\| \leqslant V_3(0)\,e^{CT} + \epsilon^{\frac{1}{6}}(e^{CT}-1).$$

As before, the triangle inequality gives

$$V_3(0) \leqslant \|g-g_\epsilon\|_3 + \|g-g_\delta\|_3 + \delta^{\frac{1}{2}}\|g-g_\epsilon\|_4 + \delta^{\frac{1}{2}}\|g-g_\delta\|_4,$$

which tends to zero as $\epsilon \downarrow 0$, $\delta < \epsilon$, by lemma 5. Thus $\|w_{xxx}\| \to 0$ as $\epsilon \downarrow 0$, for any finite $T > 0$.

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For $s > 3$, the proposition is proved by an inductive argument similar in structure to the argument just given for $s = 3$. Since $s > 3$, it follows from the remark following (5.17) that

$$\|w\|_2 \leqslant C\epsilon^{\frac{1}{3}}, \quad \text{whence} \quad |w|,\ |w_x| \leqslant C\epsilon^{\frac{1}{3}},$$

as $\epsilon \downarrow 0$. For $j < s-1$, assume inductively that

$$\|w\|_{j-1} \leqslant C\epsilon^{\frac{1}{3}} \quad \text{as} \quad \epsilon \downarrow 0. \tag{5.20}$$

Then (5.20) holds with $j-1$ replaced by j. To prove this, use (5.11) as before, setting

$$V_j(t)^2 = \int_{-\infty}^{\infty} [w_{(j)}^2 + \delta w_{(j+1)}^2]\, \mathrm{d}x. \tag{5.21}$$

Then (5.11) is expressed exactly as

$$V_j(t)^2 = V_j(0)^2 - 2\int_0^t \int_{-\infty}^{\infty} [(uw + \tfrac{1}{2}w^2)_{(j+1)} - (\epsilon - \delta)\, u_{t,(j+2)}]\, w_{(j)}\, \mathrm{d}x\, \mathrm{d}\tau. \tag{5.22}$$

An estimate of the integral on the right hand side of (5.22) is required. First use Leibnitz' rule.

$$\begin{aligned} \tfrac{1}{2}I &= -\int_0^t \int_{-\infty}^{\infty} [(uw + \tfrac{1}{2}w^2)_{(j+1)} - (\epsilon-\delta)\, u_{t,(j+2)}]\, w_{(j)}\, \mathrm{d}x\, \mathrm{d}\tau \\ &= -\int_0^t \int_{-\infty}^{\infty} \left(\sum_{k=0}^{j+1} c_k w_{(j+1-k)} w_{(j)} u_{(k)} + \sum_{k=0}^{j+1} c_k w_{(j+1-k)} w_{(j)} w_{(k)} - (\epsilon-\delta)\, u_{t,(j+2)} w_{(j)} \right) \mathrm{d}x\, \mathrm{d}\tau. \end{aligned} \tag{5.23}$$

Separate out the top derivative terms and estimate the rest directly.

$$\begin{aligned} \tfrac{1}{2}I \leqslant{} & C\int_0^t \int_{-\infty}^{\infty} \left(\sum_{k=1}^{j+1} |w_{(j+1-k)} w_{(j)} u_{(k)}| + \sum_{k=1}^{j} |w_{(j+1-k)} w_{(k)} w_{(j)}| + \epsilon |u_{t,(j+2)} w_{(j)}| \right) \mathrm{d}x\, \mathrm{d}\tau \\ & - \int_0^t \int_{-\infty}^{\infty} (w_{(j+1)} w_{(j)} u + 2w w_{(j)} w_{(j+1)})\, \mathrm{d}x\, \mathrm{d}\tau. \end{aligned}$$

From the induction hypothesis (5.20), $\|w\|_{j-1} \leqslant C\epsilon^{\frac{1}{3}}$, so that for $0 \leqslant k \leqslant j-2$, $|w_{(k)}| \leqslant C\epsilon^{\frac{1}{3}}$ on $\mathbb{R} \times [0, T]$. From corollary 1, $\|u\|_s \leqslant C$ and $\|v\|_s \leqslant C$ so $|u_{(k)}| \leqslant C$ and $|w_{(k)}| \leqslant C$ on $\mathbb{R} \times [0, T]$ so long as $0 \leqslant k \leqslant s-1$. From corollary 2, $\epsilon^{\frac{1}{2}}\|u_t\|_s \leqslant C$. Thus since $j+1 < s$, we may assemble these facts and conclude

$$\tfrac{1}{2}I \leqslant C\int_0^t (\|w_{(j)}\|^2 + \epsilon^{\frac{1}{3}}\|w_{(j)}\|)\, \mathrm{d}\tau - \int_0^t \int_{-\infty}^{\infty} (u w_{(j)} w_{(j+1)} + 2w w_{(j)} w_{(j+1)})\, \mathrm{d}x\, \mathrm{d}\tau.$$

Integrating by parts, the second integral is expressible in the form

$$\int_0^t \int_{-\infty}^{\infty} [(\tfrac{1}{2}u_x + w_x)\, w_{(j)}^2]\, \mathrm{d}x\, \mathrm{d}\tau \leqslant C\int_0^t \|w_{(j)}\|^2\, \mathrm{d}\tau.$$

Putting together the pieces gives

$$V_j(t)^2 \leqslant V_j(0)^2 + 2C\int_0^t [V_j(\tau)^2 + \epsilon^{\frac{1}{3}} V_j(\tau)]\, \mathrm{d}\tau, \tag{5.24}$$

from which, on $[0, T]$, one concludes

$$\|w_{(j)}\| \leqslant V_j(t) \leqslant V_j(0)\, \mathrm{e}^{CT} + \epsilon^{\frac{1}{3}}(\mathrm{e}^{CT} - 1). \tag{5.25}$$

From lemma 5,

$$\begin{aligned} V_j(0) &\leqslant \|g - g_\epsilon\|_j + \|g - g_\delta\|_j + \delta^{\frac{1}{2}}\|g - g_\epsilon\|_{j+1} + \delta^{\frac{1}{2}}\|g - g_\delta\|_{j+1} \\ &\leqslant C\epsilon^{\frac{1}{3}}. \end{aligned} \tag{5.26}$$

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Thus $\|w\|_j \leqslant C\epsilon^{\frac{1}{3}}$ as required. The inductive step being confirmed, there follows that

$$\|w\|_{s-2} \leqslant C\epsilon^{\frac{1}{3}}. \tag{5.27}$$

The argument just given now holds line for line for $j = s-1$, except that in (5.26) one only obtains the bound $C\epsilon^{\frac{1}{6}}$ from lemma 5. The conclusion for $j = s-1$ is therefore

$$\|w\|_{s-1} \leqslant C\epsilon^{\frac{1}{6}}. \tag{5.28}$$

Finally, for $j = s$, $\epsilon^{\frac{1}{6}}\|u\|_{s+1}$ and $\epsilon^{\frac{5}{6}}\|u_{t,(s+2)}\|$ are bounded. Hence proceeding as above, the following estimate obtains

$$\|w_{(s)}\| \leqslant (\|g-g_\epsilon\|_s + \|g-g_\delta\|_s + C\epsilon^{\frac{1}{3}})\,\mathrm{e}^{CT} + \epsilon^{\frac{1}{6}}(\mathrm{e}^{CT}-1), \tag{5.29}$$

which, owing again to lemma 5, converges to zero as $\epsilon \downarrow 0$. This finishes the proof of the proposition.

Remark. Note again, for later reference, that the various constants appearing in the proof of proposition 5 depend only on T and on $\|g\|_k$ (where $k \leqslant s$ depends on which constant is in question) and are independent of sufficiently small ϵ.

Corollary. The functions $u_t(x,t,\epsilon)$ are Cauchy in $\mathscr{H}_T^{s-3}$ as $\epsilon \downarrow 0$.

Proof. Again suppose $\delta \leqslant \epsilon$ and let $u = u_\epsilon$, $v = u_\delta$ and $w = u-v$. Then as in (5.11),

$$w_t = -(uw + \tfrac{1}{2}w^2)_x - w_{xxx} + \delta w_{xxt} + (\epsilon-\delta)\,u_{xxt}. \tag{5.30}$$

The convergence to 0 in $\mathscr{H}_T^{s-3}$ of the last two terms on the right side of (5.30) as $\epsilon \downarrow 0$ is guaranteed by corollary 2. The convergence to 0 of the other two terms on the right of (5.30) in $\mathscr{H}_T^{s-3}$ as $\epsilon \downarrow 0$ is immediate from the last proposition.

The bits and pieces needed to prove our main existence theorem for the initial-value problem for K.-dV. have now been assembled. (Existence under weaker hypotheses on the initial data is considered in appendix A.)

Theorem 1. Let $g \in H^s$, where $s \geqslant 3$. Then there exists a unique solution u, which is in $\mathscr{H}_T^s$ for all finite $T > 0$, to the K.-dV. initial-value problem with initial data g.

Proof. Uniqueness is quite easy, as has been pointed out by Sjöberg (1967, 1970) first, and by nearly all the mathematical papers on the subject since. If there were two such solutions u and v, then defining $w = u-v$, it is immediate that w satisfies the initial-value problem

$$w_t + \tfrac{1}{2}[(u+v)\,w]_x + w_{xxx} = 0, \quad w(x,0) \equiv 0. \tag{5.31}$$

Multiply (5.31) by w and integrate over $\mathbb{R}$ to gain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} w^2\,\mathrm{d}x = \int_{-\infty}^{\infty}(u+v)\,ww_x\,\mathrm{d}x = -\frac{1}{2}\int_{-\infty}^{\infty}(u_x+v_x)\,w^2\,\mathrm{d}x$$
$$\leqslant C\int_{-\infty}^{\infty} w^2(x,t)\,\mathrm{d}x.$$

It follows from Gronwall's theorem that

$$\int_{-\infty}^{\infty} w^2(x,t)\,\mathrm{d}x = 0 \quad \text{for all} \quad t \geqslant 0.$$

Hence $w = 0$ almost everywhere, and since w is continuous, w is zero everywhere. The boundary terms in the above integrations by parts all vanish since $u, v \in H^3$ at least, for each $t \geqslant 0$.

Existence is not difficult in the light of the present machinery. Let g_ϵ denote the regularization of g defined in (5.1), and u_ϵ the corresponding solution to the regularized initial-value problem (5.2). Then from the results of proposition 5 and its corollary, for each finite $T > 0$, as $\epsilon \downarrow 0$,

$$\left.\begin{aligned} u_\epsilon &\to u \quad \text{in} \quad \mathscr{H}^s_T, \\ \partial_t u_\epsilon &\to v \quad \text{in} \quad \mathscr{H}^{s-3}_T. \end{aligned}\right\} \tag{5.32}$$

From this it is immediate that

$$\left.\begin{aligned} \partial_x(u_\epsilon^2) &\to \partial_x(u^2) \quad \text{in} \quad \mathscr{H}^{s-1}_T, \\ \partial_{xxx} u_\epsilon &\to \partial_{xxx} u \quad \text{in} \quad \mathscr{H}^{s-3}_T. \end{aligned}\right\} \tag{5.33}$$

Further, $\partial_t u_\epsilon$ is bounded in $\mathscr{H}^{s-3}_T$, so $\partial_x^2 \partial_t u_\epsilon$ is bounded in $\mathscr{H}^{s-5}_T$. It follows that at least in the sense of distributions,

$$\epsilon\, \partial_x^2 \partial_t u_\epsilon \to 0 \quad \text{in} \quad \mathscr{D}'. \tag{5.34}$$

(5.32) implies that $u_\epsilon \to u$ in the distribution sense, so $\partial_t u_\epsilon \to \partial_t u$ in the distribution sense and hence $v = u_t$. Combining this with (5.32)–(5.34), shows that at least in the sense of distributions,

$$u_t + u u_x + u_{xxx} = 0, \quad u(x, 0) = g(x). \tag{0.35}$$

The initial data being correctly taken on is a consequence of lemma 5 and (5.32). Since $u \in \mathscr{H}^s_T$ and $u_t \in \mathscr{H}^{s-3}_T$, u is seen to be an L_2-solution of the initial-value problem (5.35) for K.–dV. if $s = 3$, and a classical solution in case $s > 3$. (The term L_2-solution connotes that all the derivatives expressed in the differential equation are, for each t, L_2-functions of the spatial variable x and the equation is satisfied for each t, almost everywhere in x.)

The choice of $T > 0$ was arbitrary. The larger T, the smaller ϵ must be in order that the bounds derived in § 3 be valid. Since interest is focused only on the limit $\epsilon \downarrow 0$ however, T may be chosen arbitrarily large and the same results still hold. Hence a *global* solution (solution of the initial-value problem on $\mathbb{R} \times [0, \infty)$) of (5.35) can be defined in the following simple fashion. Let u_K be the solution of the initial-value problem (5.35) on $\mathbb{R} \times [0, K]$, for $K = 1, 2, \ldots$. The uniqueness result shows that if $L > K$, then $u_L|_{[0,K]} = u_K$. Therefore define a function u on $\mathbb{R} \times [0, \infty)$ by $u(x, t) = u_K(x, t)$ for $t \leqslant K$. u is then well defined and provides a global solution to the K.–dV. initial-value problem, which lies in $\mathscr{H}^s_T$ for all finite $T > 0$ by its construction. This finishes the proof of the theorem.

The solutions guaranteed in theorem 1 have more regularity properties than stated above. This will be elucidated presently. Consider now the question of how many of the formal polynomial invariants (conservation laws) of the K.–dV. equation found by Miura *et al.* (1968) are actually constants of the motion of the solutions guaranteed above. A conservation law of K.–dV. generally is a functional I which maps some function class in the spatial variable (e.g. H^s) to the real numbers such that if $u(x, t)$ is a solution of K.–dV. which is, for each $t \geqslant 0$, a member of the function class on which I acts, then $I(u)$ is in fact independent of t. As a simple example, consider a solution u of K.–dV. corresponding to initial data $g \in H^3$, as guaranteed by theorem 1. Then

$$I_0(u) = \|u\|^2 = \int_{-\infty}^{\infty} u^2(x, t)\, \mathrm{d}x,$$

is a conservation law. To see this differentiate I_0 with respect to time. Elementary real variable theory is used to justify the following computation.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_0(u) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u(x,t)^2\,\mathrm{d}x = 2\int_{-\infty}^{\infty} uu_t\,\mathrm{d}x = -2\int_{-\infty}^{\infty} u(uu_x+u_{xxx})\,\mathrm{d}x \\ &= -2[\tfrac{1}{3}u^3+uu_{xx}]_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} u_x u_{xx}\,\mathrm{d}x \\ &= -2[\tfrac{1}{3}u^3+uu_{xx}-\tfrac{1}{2}u_x^2]_{x=-\infty}^{x=\infty} = 0.\end{aligned}$$

The conclusion is therefore that $I_0(u)$ is a constant, independent of time. There are a countable infinity of such invariants, taking the form (Kruskal *et al.* 1970, theorem 6)

$$I_k(u) = \int_{-\infty}^{\infty} [u_{(k)}^2 - c_k uu_{(k-1)}^2 + Q_k(u, \ldots, u_{(k-2)})]\,\mathrm{d}x, \tag{5.36}$$

for each $k = 0, 1, 2, \ldots$, where Q_k is a polynomial of 'rank' $k+2$. Here the definition of rank given by Miura *et al.* (1968) is followed, letting a monomial $u_{(0)}^{a_0} u_{(1)}^{a_1} \ldots u_{(p)}^{a_p}$ have rank $\sum_{i=0}^{p}(1+\frac{1}{2}i)\,a_i$, and then defining the rank of a polynomial to be the maximum of the ranks of its monomial summands. In fact, Q_k is composed entirely of monomials of rank $k+2$. If one of these nonlinear functionals I_k is indeed a constant of the motion of solutions of K.–dV., useful *a priori* estimates on the behaviour of solutions can be deduced, which hold for *all* $t \geqslant 0$. For example, if $I_0(u)$ is independent of time, it follows trivially that $\|u\|$ is bounded for all $t \geqslant 0$. More generally, if it is known that $I_0, I_1, \ldots, I_k$, when evaluated at a solution u of K.–dV., are all independent of time, and the initial data lies in H^k, then it follows easily from the results of proposition 1 that $\|u\|_k$ is bounded for all $t \geqslant 0$. This is most easily seen by induction on k, the case $k = 0$ already in hand. If the claim holds for $k-1$, then suppose $I_0(u), \ldots, I_k(u)$ are independent of time. By the induction hypothesis, $\|u\|_{k-1}$ is bounded independent of $t \geqslant 0$. By proposition 1, it follows that: (i) $\|u\|, \ldots, \|u_{(k-1)}\|$ and (ii) $|u|, \ldots, |u_{(k-2)}|$ are all bounded, independently of $t \geqslant 0$ and of $x \in \mathbb{R}$. Then from (5.36), for any $t \geqslant 0$, if $I_k(u) \equiv C$,

$$\int_{-\infty}^{\infty} u_{(k)}^2\,\mathrm{d}x = C + c_k\int_{-\infty}^{\infty} uu_{(k-1)}^2\,\mathrm{d}x - \int_{-\infty}^{\infty} Q_k(u, \ldots, u_{(k-2)})\,\mathrm{d}x. \tag{5.37}$$

The right side of (5.37) is easily bounded independent of $t \geqslant 0$ for $k > 1$ from (i) and (ii) above. For $k = 1$, (5.37) takes the form

$$\int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x = C + \frac{1}{3}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x,$$

and the argument of proposition 2 in §4 can be applied to obtain time independent bounds. These results are summarized in the next proposition.

Proposition 6. Let u be a solution of the K.–dV. equation on $\mathbb{R}\times[0,\infty)$ which is in H^k for each fixed $t \geqslant 0$ and suppose $I_0(u), \ldots, I_k(u)$ are invariant with t. Then $\|u\|_k$ is bounded uniformly for all $t \geqslant 0$.

Thus it is of interest to determine how many of the invariants (5.36) are available. Clearly if the initial data g is not in H^s, we cannot have all of $I_0, \ldots, I_s$ invariant, for at $t = 0$ at least one of $I_0(g), \ldots, I_s(g)$ is not a convergent integral. Direct verification of the conservation laws, as outlined above for $I_0(u)$ assuming $g \in H^3$, can be justified only for $k \leqslant s-3$ (see Benjamin *et al.* (1972, section 2)). Indirect means prove the best possible result, however, as is shown in the next theorem.

THEOREM 2. Let $g \in H^s$, $s \geqslant 3$, and let u be the solution of the initial value problem (5.35) for the K.–dV. equation guaranteed by theorem 1. Then $I_0(u), \ldots, I_s(u)$ are independent of time.

Proof. Let $k \leqslant s$. For the regularized initial-value problem with smoothed data (5.2), the identity

$$J_k(u) = J_k(g_\epsilon) + \epsilon \int_0^t \int_{-\infty}^{\infty} \left\{ \sum_{j=0}^{k-2} \frac{\partial Q_k}{\partial u_{(j)}} u_{t,(j+2)} + c_k [2u_x u_{(k-1)} u_{t,(k)} - u_{xxt} u_{(k-1)}^2 - u_t u_{(k)}^2] \right\} \mathrm{d}x \, \mathrm{d}\tau, \tag{5.38}$$

is valid, where

$$J_k(u) = \int_{-\infty}^{\infty} [(1 - \epsilon c_k u) u_{(k)}^2 + \epsilon u_{(k+1)}^2 - c_k u u_{(k-1)}^2 + Q_k] \, \mathrm{d}x. \tag{5.39}$$

The subscript ϵ in u_ϵ has been dropped for convenience of writing. The identity (5.38) is derived, from the fact that u is a C^∞ function of both variables all of whose derivatives are in H^∞ for each fixed $t \geqslant 0$, by differentiating $I_k(u)$ with respect to t. This would give 0 if u satisfied K.–dV., but since u is instead a solution of the regularized K.–dV. equation, there is a remainder, formally of order ϵ. After appropriate integrations by parts, this is thrown into the form (5.38)–(5.39). It follows directly from lemma 5 at the beginning of this section that

$$J_k(g_\epsilon) \to I_k(g) \quad \text{as} \quad \epsilon \downarrow 0. \tag{5.40}$$

It follows from corollary 1 to lemma 5 and the result of proposition 5 that

$$J_k(u_\epsilon) \to I_k(u) \quad \text{as} \quad \epsilon \downarrow 0, \tag{5.41}$$

where u is the solution of the initial-value problem (5.35) for the K.–dV. equation as in theorem 1. Finally, using corollary 1 and corollary 2 to lemma 5, one deduces that the integral on the right side of (5.38) converges to 0 at least at the rate $\epsilon^{\frac{1}{2}}$ as $\epsilon \downarrow 0$. Combining this with (5.40–5.41), (5.38) yields, in the limit $\epsilon \downarrow 0$,

$$I_k(u) = I_k(g), \tag{5.42}$$

and since $t \geqslant 0$ and $k \leqslant s$ were arbitrary in this calculation, this is the required invariance result.

Proposition 6 may be combined with theorems 1 and 2, and the analysis in the proof of the corollary to proposition 5 applied to higher time derivatives, to derive the final result in this section.

THEOREM 3. Let $g \in H^s$ where $s \geqslant 3$. Then there exists a unique global solution u of the initial-value problem (5.35) for the K.–dV. equation which lies in $\mathscr{H}^s_\infty$. Furthermore, if $s - 3l \geqslant 0$, $\partial_t^l u \in \mathscr{H}^{s-3l}_\infty$. (In the notation introduced in (2.4), $u \in \mathscr{X}_{s,\infty}$.)

6. Continuous dependence of solutions on the initial data

The main result of this section is a result which, when combined with theorem 3 shows that the initial-value problem for K.–dV. is well posed in Hadamard's classical sense. Let U: $H^s \to \mathscr{X}_{s,\infty}$ be the mapping which assigns to $g \in H^s$ the unique solution u of the K.–dV. equation with initial data g. The continuous dependence result then states roughly that U is a continuous mapping.

Before writing the precise theorem, a comment is deserved concerning what *cannot* be proved. It cannot be shown that U: $H^s \to \mathscr{X}_{s,\infty}$ is continuous, as one can see by a simple counter-example, given already in Benjamin *et al.* (1972) for the alternative equation (1.1). Specifically, there exists, for each $C > 0$, a similarity solution $\phi = \phi_C$ of the Korteweg–de Vries equation, known already to Korteweg and de Vries in 1895. This solution is called a *solitary wave* solution of K.–dV.

and was inspired by Scott Russell's (1844) experimental work on water waves in channels. The solution has the form

$$u_C(x,t) = \phi_C(x-Ct), \tag{6.1}$$

where

$$\phi_C(z) = 3C\,\mathrm{sech}^2\,(\tfrac{1}{2}C^{\frac{1}{2}}z). \tag{6.2}$$

Of course an arbitrary translate of ϕ_C is also a smooth solution of K.–dV. Elementary estimates show that for any $s \geqslant 0$,

$$\phi_C \to \phi_D \quad \text{in} \quad H^s \quad \text{as} \quad C \to D \quad \text{in} \quad \mathbb{R}. \tag{6.3}$$

However, owing to their differing speeds of propagation, the norm of the difference $\|u_C - u_D\|_s$ of the associated solutions to the initial-value problem (5.31) for K.–dV. has

$$\lim_{t\to\infty} \|u_C - u_D\|_s = \|u_C\|_s + \|u_D\|_s. \tag{6.4}$$

Thus u_C does not converge to u_D in H^s uniformly over *all* times. Hence the impossibility of proving results valid uniformly in time is seen explicitly, at least in this simple frame of reference. It deserves remark that stability over the unbounded time interval for the solitary wave solution to K.–dV. with respect to a different metric, which picks out the 'shape' of the wave, has been demonstrated by Benjamin (1972) (see also Bona 1975).

THEOREM 4. Let $T > 0$ be given, and let $\boldsymbol{U}: H^s \to \mathscr{X}_{s,T}$ be the restriction to the time interval $[0, T]$ of the map assigning to $g \in H^s$, $s \geqslant 3$, the unique global solution u of (5.35) for initial data g. Then $\boldsymbol{U}$ is continuous.

Proof. Remark first that it is enough to prove that $\boldsymbol{U}: H^s \to \mathscr{H}^s_T$ continuously. For it will then follow inductively from the differential equation that $\boldsymbol{U}: H^s \to \mathscr{X}_{s,T}$ continuously. For example, if $s = 3$, and $\boldsymbol{U}: H^3 \to \mathscr{H}^3_T$ is continuous, then $\boldsymbol{U}: H^3 \to \mathscr{H}^{0,1}_T$ is continuous. For this it is enough to see that the mapping $\boldsymbol{V}: H^3 \to \mathscr{H}_T$ given by $\boldsymbol{V}(g) = \partial_t \boldsymbol{U}(g)$ is continuous, since it is known *a fortiori* that $g \to u$ is continuous from H^3 to $\mathscr{H}_T$. But if $g, h \in H^3$ and $u = \boldsymbol{U}(g)$, $v = \boldsymbol{U}(h)$ are the associated solutions of the K.–dV. initial-value problems posed with initial data g and h respectively, then again by using the elementary inequality (4.5),

$$\begin{aligned}
\|u_t - v_t\| &\leqslant \|uu_x + u_{xxx} - vv_x - v_{xxx}\| \\
&\leqslant \|uu_x - vv_x\| + \|u_{xxx} - v_{xxx}\| \\
&\leqslant \|(u-v)\,u_x\| + \|(u_x - v_x)\,v\| + \|u-v\|_3 \\
&\leqslant \|u-v\|_1\,\|u_x\| + \|u_x - v_x\|\,\|v\|_1 + \|u-v\|_3 \\
&\leqslant (\|u\|_1 + \|v\|_1 + 1)\,\|u-v\|_3.
\end{aligned} \tag{6.5}$$

Taking the supremum over $t \in [0, T]$ yields

$$\|\boldsymbol{V}(g) - \boldsymbol{V}(h)\|_{\mathscr{H}_T} = \|u_t - v_t\|_{\mathscr{H}_T} \leqslant (\|u\|_{\mathscr{H}^1_T} + \|v\|_{\mathscr{H}^1_T} + 1)\,\|\boldsymbol{U}(g) - \boldsymbol{U}(h)\|_{\mathscr{H}^3_T}, \tag{6.6}$$

which shows $\boldsymbol{V}: H^3 \to \mathscr{H}_T$ is continuous since $\boldsymbol{U}: H^3 \to \mathscr{H}^3_T$ is known to be continuous.

To show $\boldsymbol{U}: H^s \to \mathscr{H}^s_T$ is continuous, let $g_n \to g$ in H^s, where $s \geqslant 3$ and let $u^n = \boldsymbol{U}(g_n)$ and $u = \boldsymbol{U}(g)$ be the associated solutions of the K.–dV. initial-value problem. It is required to show that $u^n \to u$ in $\mathscr{H}^s_T$, or what is the same, $\|u^n - u\|_s \to 0$ as $n \to \infty$, uniformly for t in $[0, T]$. Let $\gamma > 0$ be given. We wish to find N so that if $n \geqslant N$, $\|u^n - u\|_s \leqslant \gamma$ for all t in $[0, T]$. By the triangle inequality,

$$\|u^n - u\|_s \leqslant \|u^n - u^n_\epsilon\|_s + \|u^n_\epsilon - u_\epsilon\|_s + \|u_\epsilon - u\|_s. \tag{6.7}$$

Here u_ϵ is the solution of the regularized initial-value problem (5.2) with smoothed data g_ϵ as in (5.1) and similarly for u_ϵ^n.

Combining the estimates (5.28) and (5.29) of proposition 5, it appears that for t in $[0, T]$ and $\delta \leqslant \epsilon$

$$\|u_\delta - u_\epsilon\|_s \leqslant C\epsilon^{\frac{1}{6}} + C(\|g-g_\epsilon\|_s + \|g-g_\delta\|_s).$$

Let $\delta \downarrow 0$ in the last inequality. Since $u_\delta \rightarrow u$ in $\mathscr{H}_T^s$ by theorem 1, it follows that, for t in $[0, T]$,

$$\|u-u_\epsilon\|_s \leqslant C\epsilon^{\frac{1}{6}} + C\|g-g_\epsilon\|_s, \tag{6.8}$$

and similarly

$$\|u^n - u_\epsilon^n\|_s \leqslant C\epsilon^{\frac{1}{6}} + C\|g_n - g_{n\epsilon}\|_s. \tag{6.9}$$

We are justified in using the same constants C in both (6.8) and (6.9) since, as remarked earlier following the proof of proposition 5, the constants appearing in the proof of proposition 5 depend only on T and on $\|g\|_s$. Since $g_n \rightarrow g$ in H^s, of course $\|g_n\|_s \leqslant M$, for some $M > 0$, for all n and hence the various constants are bounded above, and C in (6.8) and (6.9) is taken to denote their supremum.

Now apply the fact, proven in lemma 5, that if $g_n \rightarrow g$ in H^s, then $\|g_n - g_{n\epsilon}\|_s$, $n = 1, 2, \ldots$, and $\|g-g_\epsilon\|_s$ all converge uniformly to zero as $\epsilon \downarrow 0$. It follows from this observation and the inequalities (6.8) and (6.9) that

$$\left.\begin{aligned} \|u^n - u_\epsilon^n\|_s &\rightarrow 0 \\ \|u - u_\epsilon\|_s &\rightarrow 0 \end{aligned}\right\} \quad \text{uniformly for } t \text{ in } [0, T] \text{ and } n = 1, 2, \ldots, \text{ as } \epsilon \downarrow 0. \tag{6.10}$$

Therefore ϵ may be chosen so small that, for all t in $[0, T]$ and all $n = 1, 2, \ldots$,

$$\|u-u_\epsilon\|_s \leqslant \tfrac{1}{3}\gamma \quad \text{and} \quad \|u^n - u_\epsilon^n\|_s \leqslant \tfrac{1}{3}\gamma. \tag{6.11}$$

Thus in order to show that for n sufficiently large $\|u^n - u\|_s \leqslant \gamma$, for t in $[0, T]$, it is only necessary to show that $\|u_\epsilon^n - u_\epsilon\|_s \rightarrow 0$ as $n \rightarrow \infty$, where $\epsilon > 0$ is fixed, but small enough that (6.11) holds. For if N is then chosen so large that for $n \geqslant N$, $\|u_\epsilon^n - u_\epsilon\|_s \leqslant \frac{1}{3}\gamma$, it then follows from (6.7) and (6.11) that for $n \geqslant N$, $\|u^n - u\|_s \leqslant \gamma$.

There is no shortage of ways to accomplish this last task. One method is to make an argument very similar to the argument given in the proof of proposition 5. However, since $\epsilon > 0$ is *fixed* for the purposes at hand, it is somewhat easier to transform the problem. Specifically, by definition, u_ϵ is the solution of the initial-value problem

$$u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0, \quad u(x, 0) = g_\epsilon(x), \tag{6.12}$$

and similarly for u_ϵ^n. But ϵ is fixed, and hence the transformation (3.2) sends (6.12) to the initial-value problem

$$v_t + v_x + vv_x - v_{xxt} = 0, \quad v(x, 0) = h(x) = \epsilon g_\epsilon(\epsilon^{\frac{1}{2}}x). \tag{6.13}$$

Define $h_n(x) = \epsilon g_{n\epsilon}(\epsilon^{\frac{1}{2}}x)$, and $h(x) = \epsilon g_\epsilon(\epsilon^{\frac{1}{2}}x)$, and let v^n and v be the solutions to the initial-value problem (6.13) posed for h_n and h respectively. Then, if $v^n \rightarrow v$ in $\mathscr{H}_R^s$, where $R > 0$ is arbitrary but finite, it follows by inverting the transformation (3.2) that $u_\epsilon^n \rightarrow u_\epsilon$ in $\mathscr{H}_T^s$.

Since $g_n \rightarrow g$ in H^s, certainly $g_n \rightarrow g$ in L_2. Estimates in Fourier transformed variables show immediately that $g_{n\epsilon} \rightarrow g_\epsilon$ in H^r for all $r \geqslant 0$. Of course the rates of convergence in the various H^r norms depend strongly on ϵ, but ϵ is fixed for now. Hence h_n, $h \in H^\infty$ and $h_n \rightarrow h$ in H^r for all $r \geqslant 0$. Lemma 2 assures us that v^n, $v \in \mathscr{H}_R^{\infty, \infty}$ for all $R > 0$. Therefore the following simple calculations are valid.

Let $w^n = v^n - v$. Then w^n satisfies the initial-value problem

$$\left.\begin{aligned} w_t^n + w_x^n + w^n w_x^n + (vw^n)_x - w_{xxt}^n &= 0, \\ w^n(x,0) = h_n(x) - h(x) &= f_n(x), \end{aligned}\right\} \tag{6.14}$$

where $f_n \to 0$ in H^r for all $r \geqslant 0$. We drop the superscript n during the following computations, for ease of writing. In analogy with (5.21), define

$$W_j(t) = \int_{-\infty}^{\infty} [w_{(j)}^2 + w_{(j+1)}^2]\,\mathrm{d}x. \tag{6.15}$$

Then multiply (6.14) by $w_{(2j)}$ and integrate over $\mathbb{R}$ and over $[0,t]$, to come to the following relation, after appropriate integrations by parts and applications of Fubini's theorem

$$W_j(t) = W_j(0) + (-1)^{j+1}\, 2 \int_0^t \int_{-\infty}^{\infty} [ww_x + (vw)_x]\, w_{(2j)}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.16}$$

Integrating by parts and applying Leibnitz' rule as in the proof of proposition 5 following (5.23), the inequality

$$W_j(t) \leqslant W_j(0) + C\left|\int_0^t \int_{-\infty}^{\infty} \left(\sum_{k=0}^{j+1} w_{(j+1-k)} w_{(k)} w_{(j)} + \sum_{k=0}^{j+1} w_{(j+1-k)} v_{(k)} w_{(j)}\right) \mathrm{d}x\,\mathrm{d}\tau\right|, \tag{6.17}$$

is derived. Here C is just twice the supremum of the binomial coefficients appearing from the use of Leibnitz' rule, and depends only on j. In case $j = 0$, this comes to

$$\begin{aligned} W_0(t) &\leqslant W_0(0) + C\left|\int_0^t \int_{-\infty}^{\infty} v_x w^2\,\mathrm{d}x\,\mathrm{d}\tau\right| \leqslant W_0(0) + C\int_0^t \int_{-\infty}^{\infty} w^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant W_0(0) + C\int_0^t W_0(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{6.18}$$

where $\sup|v_x|$ has been pulled outside the integral. (6.18) implies

$$W_0(t) \leqslant W_0(0)\,\mathrm{e}^{Ct}. \tag{6.19}$$

Reinterpreting this, it is implied that

$$\|w^n\|_1 \leqslant \|f_n\|_1\,\mathrm{e}^{CR}, \tag{6.20}$$

from which it is obvious that $\|w^n\|_1 \to 0$ as $n \to \infty$, uniformly on $[0,R]$. Now assuming inductively that $\|w^n\|_j \to 0$ as $n \to \infty$ uniformly on $[0,R]$, one can derive that $\|w^n\|_{j+1} \to 0$ as $n \to \infty$ by using (6.17) to derive an inequality of the form

$$W_j(t) \leqslant W_j(0) + C\int_0^t [W_j(\tau) + a_n W_j(\tau)^{\frac{1}{2}}]\,\mathrm{d}\tau,$$

where $a_n \to 0$ as $n \to \infty$. It follows that for t in $[0,R]$

$$W_j(t)^{\frac{1}{2}} \leqslant W_j(0)^{\frac{1}{2}}\,\mathrm{e}^{CR} + a_n(\mathrm{e}^{CR} - 1), \tag{6.21}$$

and this is enough to conclude $\|w^n\|_{j+1} \to 0$ as $n \to \infty$ uniformly for t in $[0,R]$. Hence it is demonstrated that $\|w^n\|_r \to 0$ as $n \to \infty$ uniformly for t in $[0,R]$, for all $r \geqslant 0$. That is $\|v^n - v\|_r \to 0$ uniformly on bounded time intervals, for all $r \geqslant 0$. The proof of the theorem is now complete.

The last part of the proof of theorem 4 has in effect provided new results concerning continuous dependence of solutions on initial data for the equation (1.1) (cf. (6.13)) not contained in the work of Benjamin *et al.* (1972), which are summarized in the next theorem.

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THEOREM 5. Consider the initial-value problem

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad u(x, 0) = g(x), \tag{6.22}$$

for $x \in \mathbb{R}$ and $t \geqslant 0$. Suppose $g \in H^m$ where $m \geqslant 1$. Then there exists a unique solution u to (6.22) which is in $\mathscr{H}^m_T$ for all finite $T > 0$. Further $\partial^l_t u \in \mathscr{H}^{m+1}_T$ for all $l > 0$. Also, the solution u (resp. $\partial^l_t u$) depends continuously in $\mathscr{H}^m_T$ (resp. $\mathscr{H}^{m+1}_T$) on the initial data g in H^m, for all $T > 0$.

7. Comparison of solutions of K.–dV. and the model (1.1)

In this section the machinery previously erected is exploited to examine a relation between the initial-value problem for K.–dV. and for the model equation (1.1). It will appear that in a certain limit, under which this type of model is generally derived, the two models give solutions, corresponding to the same initial data, which are very close together at least over finite time intervals. By explicitly considering solitary-wave initial data one can show that this result is best possible in the sense that one cannot expect close agreement over the unbounded time interval.

The specific assumptions which come to the fore in the derivation of K.–dV. or (1.1) as models for surface water waves for example are that the amplitude of the wave is inversely proportional to the square of the wave length, the amplitude being small in comparison to the undisturbed depth of the fluid (cf. Peregrine 1972). This situation can be reflected for a given function g defined on $\mathbb{R}$ by considering the associated function

$$h_\epsilon(x) = \epsilon g(\epsilon^{\frac{1}{2}} x), \tag{7.1}$$

for $\epsilon \ll 1$. The question posed is how do the two models in question respond to the same initial data (7.1) when ϵ is small? Put more carefully, let $u^* = u^*(x, t; \epsilon)$ be the solution of the K.–dV. initial-value problem

$$u^*_t + u^*_x + u^* u^*_x + u^*_{xxx} = 0, \quad u^*(x, 0) = h_\epsilon(x), \tag{7.2}$$

and let $v^* = v^*(x, t; \epsilon)$ be the solution of the initial-value problem for (1.1)

$$v^*_t + v^*_x + v^* v^*_x - v^*_{xxt} = 0, \quad v^*(x, 0) = h_\epsilon(x). \tag{7.3}$$

Then we would like to know by how much u^* and v^* differ from each other over some fixed finite time interval $[0, T]$ say. From the elementary inequality (4.5),

$$\sup_{\mathbb{R} \times [0, T]} |u^*(x, t) - v^*(x, t)| \leqslant \sup_{[0, T]} \|u^* - v^*\|_1. \tag{7.4}$$

Hence an estimate for the H^1 norm of the difference of u^* and v^* will yield a pointwise estimate on their difference.

At this point it should be observed that $h_\epsilon \to 0$ in the function spaces under consideration here and therefore the continuous dependence results of § 5 for the two model equations imply that both u^* and v^* are tending to 0 as $\epsilon \downarrow 0$. Hence they are approaching each other since they are both approaching the zero function. We have in mind a more substantial result than this. The point is that the problem must be considered with the right magnifying glass in order to determine whether the above argument represents the best that can be said, or whether the two solutions approach each other faster than they approach zero.

For the purpose of comparison, it is therefore convenient to make the transformation (inverse to the transformation (3.2))

$$u(x, t) = \epsilon^{-1} u^*(\epsilon^{-\frac{1}{2}} x + \epsilon^{-\frac{3}{2}} t, \epsilon^{-\frac{3}{2}} t), \tag{7.5}$$

and similarly for v and v^*. This transforms to a coordinate system moving with the wave and scaled inversely to the scaling of h_ϵ. Hence function values of u^* and v^* are magnified and their length scale shortened leaving the following equations satisfied by u and v respectively.

$$u_t + uu_x + u_{xxx} = 0, \quad u(x, 0) = g(x), \tag{7.6}$$

and

$$v_t + vv_x + v_{xxx} - \epsilon v_{xxt} = 0, \quad v(x, 0) = g(x). \tag{7.7}$$

Clearly the problem of comparing u and v in (7.6) and (7.7) can be attacked by way of the arguments in § 5, making use of the *a priori* bounds derived in § 4. Let $w = v - u$. Then w satisfies the following initial-value problem analogous to (5.10) with $\delta = 0$ and zero initial data,

$$w_t + (vw + \tfrac{1}{2}w^2)_x + w_{xxx} - \epsilon v_{xxt} = 0, \quad w(x, 0) \equiv 0. \tag{7.8}$$

Suppose at the outset that $g \in H^\infty$ so that both u and v are C^∞ functions in their two variables and all their partial derivatives lie in $\mathscr{H}_T$. Then just as in (5.11), the identities

$$\int_{-\infty}^{\infty} w_{(j)}^2 \, dx = -2 \int_0^t \int_{-\infty}^{\infty} [(vw + \tfrac{1}{2}w^2)_{(j+1)} - \epsilon v_{t,(j+2)}] \, w_{(j)} \, dx \, d\tau, \tag{7.9}$$

are valid for $j = 0, 1, 2, \ldots$. Define

$$V_j(t)^2 = \int_{-\infty}^{\infty} w_{(j)}^2 \, dx, \tag{7.10}$$

for convenience of writing. Of course $V_j(0) = 0$ for all j, as is already reflected in formula (7.9). In order to obtain reasonably sharp results the relation (7.9) will be used in several ways. We begin with a lemma which extends the result of (4.22).

Lemma 6. *Let $g \in H^\infty$ and let v be the corresponding C^∞ solution of the regularized initial-value problem (7.7) all of whose derivatives are in $\mathscr{H}_T$ for all $T > 0$. Then for any integer $l \geq 0$ and $t \geq 0$ the following inequalities hold.*

$$\left.\begin{aligned} &\text{(i)} && \|\partial_x^l v_t\| \leq (\|v\|_{l+3} + \|v\|_{l+1}^2), \\ &\text{(ii)} && \|\partial_x^l v_t\| \leq \tfrac{1}{2}\epsilon^{-\frac{1}{2}}(\|v\|_{l+2} + \|v\|_l^2), \\ &\text{(iii)} && \|\partial_x^l v_t\| \leq \epsilon^{-1}(\|v\|_{l+1} + \|v\|_{l-1}^2). \end{aligned}\right\} \tag{7.11}$$

Proof. Write the differential equation in the form

$$(I - \epsilon \partial_x^2) v_t = -(vv_x + v_{xxx}),$$

and invert the operator $I - \epsilon \partial_x^2$ as before to obtain

$$v_t = -(I - \epsilon \partial_x^2)^{-1}(vv_x + v_{xxx}) = -(I - \epsilon \partial_x^2)^{-1}(V + v_{xxx}), \tag{7.12}$$

where $V = vv_x$. By an easy estimate, $\|V\|_k \leq \|v\|_{k+1}^2$ for all $t \geq 0$. Then of course

$$w_l = \partial_x^l v_t = -\partial_x^l (I - \epsilon \partial_x^2)^{-1}(V + v_{xxx}). \tag{7.13}$$

In Fourier transformed variables this takes the form

$$\hat{w}_l = \frac{-(ik)^l}{1 + \epsilon k^2}[\hat{V} + (ik)^3 \hat{v}], \tag{7.14}$$

where ^ denotes Fourier transforms as before. Hence

$$\|w_l\|^2 = \|\hat{w}_l\|^2 = \int_{-\infty}^{\infty} \frac{k^{2l}}{(1 + \epsilon k^2)^2} |\hat{V} + (ik)^3 \hat{v}|^2 \, dk. \tag{7.15}$$

The estimation of (7.15) is carried out in three ways corresponding to (7.11) (i), (ii) and (iii). First because $1+\epsilon k^2 \geqslant 1$ for all k,

$$\|w_l\| \leqslant \left[\int_{-\infty}^{\infty} k^{2l}\,|\hat{V}+(\mathrm{i}k)^3\hat{v}|^2\,\mathrm{d}k\right]^{\frac{1}{2}} \leqslant \|V+v_{xxx}\|_l$$
$$\leqslant \|V\|_l+\|v_{xxx}\|_l \leqslant \|v\|^2_{l+1}+\|v\|_{l+3}.$$

For (ii) proceed as follows.

$$\|w_l\| \leqslant \sup_{k\in\mathbb{R}}\left(\frac{|k|}{1+\epsilon k^2}\right)\left[\int_{-\infty}^{\infty} k^{2(l-1)}\,|\hat{V}+(\mathrm{i}k)^3\hat{v}|^2\,\mathrm{d}k\right]^{\frac{1}{2}}$$
$$\leqslant \frac{1}{2\epsilon^{\frac{1}{2}}}\|V+v_{xxx}\|_{l-1} \leqslant \frac{1}{2\epsilon^{\frac{1}{2}}}(\|V\|_{l-1}+\|v_{xxx}\|_{l-1})$$
$$\leqslant \frac{1}{2\epsilon^{\frac{1}{2}}}(\|v\|^2_l+\|v\|_{l+2}).$$

Finally for (iii),

$$\|w_l\| \leqslant \sup_{k\in\mathbb{R}}\left(\frac{k^2}{1+\epsilon k^2}\right)\left[\int_{-\infty}^{\infty} k^{2(l-2)}\,|\hat{V}+(\mathrm{i}k)^3\hat{v}|^2\,\mathrm{d}k\right]^{\frac{1}{2}}$$
$$\leqslant \frac{1}{\epsilon}\|V+v_{xxx}\|_{l-2} \leqslant \frac{1}{\epsilon}(\|V\|_{l-2}+\|v_{xxx}\|_{l-2})$$
$$\leqslant \frac{1}{\epsilon}(\|v\|^2_{l-1}+\|v\|_{l+1}).$$

This establishes the lemma.

Corollary. For v as in lemma 6, the following inequalities are valid for $l \geqslant 0$.

$$\left.\begin{aligned}
&\text{(i)} && \|\partial_x^l v_t\| \leqslant C(\|g\|_{l+4}+\|g\|^2_{l+2}) \\
& && \qquad\leqslant C(\|g\|_{l+4}+\|g\|^2_{l+4}),\\
&\text{(ii)} && \|\partial_x^l v_t\| \leqslant C\epsilon^{-\frac{1}{2}}(\|g\|_{l+3}+\|g\|^2_{l+1}) \\
& && \qquad\leqslant C\epsilon^{-\frac{1}{2}}(\|g\|_{l+3}+\|g\|^2_{l+3}),\\
&\text{(iii)} && \|\partial_x^l v_t\| \leqslant C\epsilon^{-1}(\|g\|_{l+2}+\|g\|^2_{l}) \\
& && \qquad\leqslant C\epsilon^{-1}(\|g\|_{l+2}+\|g\|^2_{l+2}),
\end{aligned}\right\} \tag{7.16}$$

holding for t in $[0, T]$ and ϵ sufficiently small, where the constants C depend on T but not on ϵ.

Proof. This is a consequence of lemma 6 and the ϵ-independent bounds obtained on solutions of the regularized initial-value problem in § 4 (see proposition 4).

Now consideration is given to the derivation of bounds on the various H^s norms of the difference $w = v-u$. Bounds will be inferred in three stages.

Proposition 8. For each $k \geqslant 0$, the difference $w = v-u$ satisfies the inequalities

$$\|w_{(k)}\| \leqslant \epsilon M_k, \tag{7.17}$$

where $M_k = M_k(T, \|g\|_{k+6})$ is independent of ϵ sufficiently small.

Proof. This proceeds easily by induction on k. To get started consider (7.9) for $j = 0$ from which the following inequality is simply derived.

$$\int_{-\infty}^{\infty} w^2\,\mathrm{d}x = V_0(t)^2 \leqslant 2 \sup_{\mathbb{R}\times[0,\,T]}|w_x+\tfrac{1}{2}v_x|\int_0^t V_0(\tau)^2\,\mathrm{d}\tau+2\epsilon\int_0^t\|v_{xxt}\|\,\|w\|\,\mathrm{d}\tau$$
$$\leqslant 2C_1\int_0^t V_0(\tau)^2\,\mathrm{d}\tau+2C_2\,\epsilon\int_0^t V_0(\tau)\,\mathrm{d}\tau, \tag{7.18}$$

where C_1 depends on a bound for $|w_x + \frac{1}{2}v_x|$ on $\mathbb{R}\times[0, T]$ and hence on T and $\|g\|_3$ by proposition 3. Likewise, C_2 depends on a bound for $\|v_{xxt}\|$ on $[0, T]$ and hence on $\|g\|_6$ by the corollary to lemma 6 above. (7.18) gives, for t in $[0, T]$,

$$\|w\| \leqslant \epsilon(C_2/C_1)\,(\mathrm{e}^{C_1 T}-1) = \epsilon M_0, \tag{7.19}$$

where M_0 depends on T and $\|g\|_6$. Now suppose that if $0 \leqslant j < k$,

$$\|w_{(j)}\| \leqslant \epsilon M_j, \tag{7.20}$$

where $k > 0$ and $M_j = M_j(T, \|g\|_{j+6})$. Use (7.9) to derive the identity (analogous to (5.22–5.23))

$$\begin{aligned} V_k(t)^2 &= -2\int_0^t\int_{-\infty}^{\infty}[(vw+\tfrac{1}{2}w^2)_{(k+1)} - \epsilon v_{t,(k+2)}]\,w_{(k)}\,\mathrm{d}x\,\mathrm{d}\tau \\ &= -2\int_0^t\int_{-\infty}^{\infty}\left[\sum_{j=0}^{k+1} c_j w_{(k+1-j)}v_{(j)}w_{(k)} + \sum_{j=0}^{k+1} c_j w_{(k+1-j)}w_{(j)}w_{(k)}\right]\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + 2\epsilon\int_0^t\int_{-\infty}^{\infty} v_{t,(k+2)}w_{(k)}\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{7.21}$$

Making use of the induction hypothesis (7.20) and of (7.16), (7.21) leads to the estimate

$$V_k(t)^2 \leqslant 2C_{2k+1}\int_0^t V_{k+1}^2(\tau)\,\mathrm{d}\tau + 2\epsilon C_{2k+2}\int_0^t V_{k+1}(\tau)\,\mathrm{d}\tau,$$

where $$C_{2k+1} = C_{2k+1}(T, \|g\|_{k+5}),$$

and $$C_{2k+2} = C_{2k+2}(T, \|g\|_{k+6}).$$

Hence for t in $[0, T]$ $$\|w_{(k)}\| = V_k(t) \leqslant \epsilon\frac{C_{2k+2}}{C_{2k+1}}(\mathrm{e}^{C_{2k+1}T}-1) = \epsilon M_k, \tag{7.22}$$

where $M_k = M_k(T, \|g\|_{k+6})$ as required.

The results of proposition 8 would already yield interesting results bearing on the problem under consideration in this section. Before stating these, another similar set of inequalities is given which can be used in conjunction with the results of proposition 8 to yield reasonably sharp results.

Proposition 9. For each $k \geqslant 0$ the difference $w = v - u$ satisfies the inequalities

$$\|w_{(k)}\| \leqslant \epsilon^{\frac{1}{2}} N_k, \tag{7.23}$$

valid for t in $[0, T]$ and ϵ sufficiently small where $N_k = N_k(T, \|g\|_{k+4})$.

Proof. Again the proof proceeds by induction on k. For $k = 0$, use (7.9) to write

$$\begin{aligned} V_0(t)^2 &\leqslant 2\sup_{\mathbb{R}\times[0,T]}|w_x + \tfrac{1}{2}v_x|\int_0^t V_0(\tau)^2\,\mathrm{d}\tau + 2\epsilon\int_0^t \|v_t\|\,\|w_{xx}\|\,\mathrm{d}\tau \\ &\leqslant C_1\int_0^t V_0(\tau)^2\,\mathrm{d}\tau + \epsilon C_2 t, \end{aligned}$$

valid for t in $[0, T]$, where C_1 depends as before on T and $\|g\|_3$ and C_2 depends also on T and $\|g\|_4$ by the corollary to lemma 6. There follows

$$\|w\|^2 = V_0(t)^2 \leqslant \epsilon(C_2/C_1)\,(\mathrm{e}^{C_1 T}-1),$$

so that $$\|w\| \leqslant \epsilon^{\frac{1}{2}} N_0, \tag{7.24}$$

for t in $[0, T]$ where $N_0 = N_0(T, \|g\|_4)$. Consider the case $k = 1$. Again use (7.9) to write

$$\begin{aligned} V_1(t)^2 &= -2\int_0^t\int_{-\infty}^{\infty} [(\tfrac{1}{2}w_x+\tfrac{3}{2}v_x)\,w_x^2+v_{xx}ww_x-\epsilon v_{xxxt}\,w_x]\,\mathrm{d}x\,\mathrm{d}\tau \\ &= -2\int_0^t\int_{-\infty}^{\infty} [(\tfrac{1}{2}w_x+\tfrac{3}{2}v_x)\,w_x^2-\tfrac{1}{2}v_{xxx}w^2+\epsilon v_t\,w_{xxxx}]\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant 2\sup_{\mathbb{R}\times[0,T]}|\tfrac{1}{2}w_x+\tfrac{3}{2}v_x|\int_0^t V_1^2(\tau)\,\mathrm{d}\tau+2\sup_{\mathbb{R}\times[0,T]}|v_{xxx}|\int_0^t V_0^2(\tau)\,\mathrm{d}\tau \\ &\quad +2\sup_{[0,T]}(\|v_t\|\,\|w_{xxxx}\|)\,\epsilon t, \end{aligned} \tag{7.25}$$

for t in $[0, T]$. As before, bound $|w_x+3v_x|$ by a constant C_3 depending on T and $\|g\|_3$. By (7.24) the middle term on the right hand side of (7.25) can be bounded by $2\epsilon t \sup_{[0,T]}\|v\|_4$ which is bounded by ϵ times a constant depending on T and $\|g\|_5$. Finally $\|v_t\|$ is bounded by a constant depending on $\|g\|_4$ and T while $\|w_{xxxx}\|$ is bounded by a constant depending on T and $\|g\|_5$ by proposition 4. In sum (7.25) yields

$$V_1^2(t) \leqslant C_3\int_0^t V_1(\tau)^2\,\mathrm{d}\tau+\epsilon C_4 t, \tag{7.26}$$

where $C_3 = C_3(T, \|g\|_3)$ and $C_4 = C_4(T, \|g\|_5)$. (7.26) now implies that for t in $[0, T]$

$$\|w_x\|^2 = V_1^2(t) \leqslant \epsilon(C_4/C_3)\,(\mathrm{e}^{C_3T}-1),$$

and hence
$$\|w_x\| \leqslant \epsilon^{\frac{1}{2}}N_1,$$

where $N_1 = N_1(T, \|g\|_5)$. The general inductive step follows lines which are by now familiar and whose details may be suitably passed over.

Finally a third method of estimation yields inequalities on $V_k(t)$ depending on T and $\|g\|_{k+3}$.

Proposition 10. For each $k \geqslant 0$ the difference $w = v-u$ satisfies inequalities of the form

$$\|w_{(k)}\| \leqslant \epsilon^{\frac{1}{4}}L_k, \tag{7.28}$$

valid for t in $[0, T]$ and ϵ sufficiently small where $L_k = L_k(T, \|g\|_{k+3})$.

Proof. Again the argument proceeds by induction on k. Here are the calculations for $k = 1$ for example. For V_1 write (7.9) in the form

$$V_1^2(t) = -2\int_0^t\int_{-\infty}^{\infty} [(\tfrac{1}{2}w_x+\tfrac{3}{2}v_x)\,w_x^2+v_{xx}ww_x-\epsilon v_{xt}\,w_{xxx}]\,\mathrm{d}x\,\mathrm{d}\tau.$$

Then if $0 \leqslant t \leqslant T$,

$$V_1(t)^2 \leqslant 2\sup_{\mathbb{R}\times[0,T]}|\tfrac{1}{2}w_x+\tfrac{3}{2}v_x|\int_0^t V_1(\tau)^2\,\mathrm{d}\tau+2t\sup_{\mathbb{R}\times[0,T]}|v_{xx}|\sup_{[0,T]}(\|w\|\,\|w_x\|)+2\epsilon t\sup_{[0,T]}(\|v_{xt}\|\,\|w_{xxx}\|).$$

Making use of (7.24) and of (7.16 ii) there follows from the last inequality

$$V_1(t)^2 \leqslant C_3\int_0^t V_1(\tau)^2\,\mathrm{d}\tau+\epsilon^{\frac{1}{2}}C_4 t,$$

where $C_3 = C_3(T, \|g\|_3)$ and similarly $C_4 = C_4(T, \|g\|_4)$. Then as before, for any $t \geqslant 0$

$$\|w_x\|^2 = V_1(t)^2 \leqslant \epsilon^{\frac{1}{2}}(C_4/C_3)\,(\mathrm{e}^{C_3t}-1),$$

so that for t in $[0, T]$,
$$\|w_x\| \leqslant \epsilon^{\frac{1}{4}}L_1, \tag{7.29}$$

where $L_1 = L_1(T, \|g\|_4)$. Again the inductive step, making use of (7.9), is close enough to previous arguments to be safely left to the reader. At the kth step, the term on the right hand side giving the most trouble is

$$2\epsilon\int_0^t\int_{-\infty}^{\infty} v_{t,(k+2)}w_{(k)}\,\mathrm{d}x\,\mathrm{d}\tau = 2\epsilon\int_0^t\int_{-\infty}^{\infty} v_{t,(k)}w_{(k+2)}\,\mathrm{d}x\,\mathrm{d}\tau,$$

which can be estimated, for t in $[0, T]$, as follows, by using proposition 4 and (7.16)

$$\begin{aligned}2\epsilon\left|\int_0^t\int_{-\infty}^{\infty} v_{t,(k)}w_{(k+2)}\,\mathrm{d}x\,\mathrm{d}\tau\right| &\leqslant 2\epsilon\int_0^t \|\partial_x^k v_t\|\,\|w_{(k+2)}\|\,\mathrm{d}\tau\\ &\leqslant 2tC\epsilon\epsilon^{-\frac{1}{2}}(\|g\|_{k+3}+\|g\|_{k+3}^2)\,\|w\|_{k+2}\\ &\leqslant \epsilon^{\frac{1}{2}}C_{k+2}t,\end{aligned}$$

where $C_{k+2} = C_{k+2}(T, \|g\|_{k+3})$. The other terms are handled by way of propositions 3 and 4, the result (7.17) of proposition 8 and the induction hypothesis.

Here is a corollary to the last three propositions which will be used to answer the query concerning the difference $w = v-u$.

Corollary. The difference $w = v-u$, where u and v are the unique smooth solutions of (7.6) and (7.7) respectively for the given data $g\in H^\infty$, satisfies the following inequalities, each valid for ϵ sufficiently small and for t in $[0, T]$ where $T > 0$ is arbitrary but finite.

$$\left.\begin{aligned}&\text{(i)}\quad \|w\|_k \leqslant \epsilon Q_k \quad \text{where}\quad Q_k = Q_k(T,\|g\|_{k+6}) \quad (k = 0, 1, \ldots),\\ &\text{(ii)}\quad \|w\|_k \leqslant \epsilon^{\frac{1}{2}}R_k \quad \text{where}\quad R_k = R_k(T,\|g\|_{k+4}) \quad (k = 0, 1, \ldots),\\ &\text{(iii)}\quad \|w\|_k \leqslant \epsilon^{\frac{1}{4}}S_k \quad \text{where}\quad S_k = S_k(T,\|g\|_{k+3}) \quad (k = 0, 1, \ldots).\end{aligned}\right\} \tag{7.30}$$

Proof. These follow from summing the first $k+1$ inequalities expressed in propositions 8, 9 and 10 respectively.

The inequalities (7.30) were derived for $g\in H^\infty$ and the resulting smooth solutions. Suppose now that g is only in H^s where $s \geqslant 3$. Approximate g in H^s by a sequence $\{g_n\}\subset H^\infty$, for example as was done in lemma 5. Letting u_n and v_n denote the respective solutions of (7.6) and (7.7) posed for the initial data g_n and setting $w_n = u_n - v_n$, it then follows from the continuous dependence results, theorems 4 and 5 in § 5, that $u_n\to u$ in $\mathscr{H}_T^s$ and $v_n\to v$ in $\mathscr{H}_T^s$. Since $g_n\to g$ in H^s, $\|g_n\|_s$ is bounded uniformly in n. Hence the various constants in (7.30) remain bounded uniformly in n and in $\epsilon\leqslant\epsilon_0$ say where ϵ_0 can be chosen independent of n by propositions 3 and 4. That is, $Q_k^n = Q_k(T,\|g_n\|_{k+6})$ is bounded uniformly in n, so long as $k\leqslant s-6$ of course, and similarly for the R_k's and S_k's with the appropriate restrictions on k. Thus letting $\bar{Q}_k = \sup_n Q_k^n$, and similarly for $\bar{R}_k$ and $\bar{S}_k$, then for all $n = 1, 2, \ldots$,

$$\left.\begin{aligned}\|w_n\|_k &\leqslant \epsilon\bar{Q}_k \quad \text{provided}\quad k\leqslant s-6,\\ \|w_n\|_k &\leqslant \epsilon^{\frac{1}{2}}\bar{R}_k \quad \text{provided}\quad k\leqslant s-4,\\ \|w_n\|_k &\leqslant \epsilon^{\frac{1}{4}}\bar{S}_k \quad \text{provided}\quad k\leqslant s-3.\end{aligned}\right\} \tag{7.31}$$

Taking the limit as $n\to\infty$ in (7.31) establishes the following result.

Proposition 11. Let $g\in H^s$ where $s\geqslant 3$ and let u and v be the $\mathscr{X}_{s,T}$ solutions of the initial-value problems (7.6) and (7.7) respectively posed with initial data g. Let $w = u-v$. Then as $\epsilon\downarrow 0$

$$\left.\begin{aligned}\|w\|_k &\leqslant \epsilon\bar{Q}_k(T,\|g\|_{k+6}) \quad (k = 0, 1, \ldots),\\ \|w\|_k &\leqslant \epsilon^{\frac{1}{2}}\bar{R}_k(T,\|g\|_{k+4}) \quad (k = 0, 1, \ldots),\\ \|w\|_k &\leqslant \epsilon^{\frac{1}{4}}\bar{S}_k(T,\|g\|_{k+3}) \quad (k = 0, 1, \ldots),\end{aligned}\right\} \tag{7.32}$$

uniformly for t in $[0, T]$, for all k such that the norms of g on the right hand side are finite.

It now remains to interpret proposition 11 in terms of $w^* = v^* - u^*$. This is done in the last result in this section.

THEOREM 6. Let $g \in H^s$ where $s \geqslant 3$, let $T > 0$ be finite and let u^* and v^* be the $\mathscr{X}_{s,T}$ solutions of the initial-value problems (7.2) and (7.3) respectively, and define $w^* = v^* - u^*$. Then

$$\left.\begin{aligned} \|w^*_{(k)}\| &\leqslant \epsilon^{\frac{1}{4}(2k+7)}\bar{Q}_k(T, \|g\|_{k+6}) \quad (k = 0, \ldots, s-6),\\ \|w^*_{(k)}\| &\leqslant \epsilon^{\frac{1}{4}(2k+5)}\bar{R}_k(T, \|g\|_{k+4}) \quad (k = 0, \ldots, s-4),\\ \|w^*_{(k)}\| &\leqslant \epsilon^{1+\frac{1}{2}k}\bar{S}_k(T, \|g\|_{k+3}) \quad (k = 1, \ldots, s-3),\end{aligned}\right\} \tag{7.33}$$

uniformly for $0 \leqslant t \leqslant T$ and ϵ sufficiently small.

Proof. This is an immediate consequence of proposition 11 and the elementary relation induced by the transformation (7.5)

$$\|v^*_{(k)}\| = \epsilon^{\frac{1}{4}(2k+3)}\|v_{(k)}\|, \quad (k = 0, 1, \ldots), \tag{7.34}$$

and similarly for u^* and u.

From theorem 6, reasonably sharp convergence results for u^* to v^* as $\epsilon \downarrow 0$ can be derived, for given L_2-smoothness of g. For example, suppose $g \in H^7$. Then both u^* and v^* are classical solutions of their respective differential equations and from theorem 6, given $T > 0$ there is a constant C depending on T and $\|g\|_7$ such that for $0 \leqslant t \leqslant T$ and ϵ small enough,

$$\|u^* - v^*\| \leqslant C\epsilon^{\frac{7}{4}} \quad \text{and} \quad \|u^*_x - v^*_x\| \leqslant C\epsilon^{\frac{9}{4}}. \tag{7.35}$$

In particular, because of (4.5),

$$\sup_{\mathbb{R}\times[0,\,T]} |u^* - v^*| \leqslant (\|u^* - v^*\|\,\|u^*_x - v^*_x\|)^{\frac{1}{2}} \leqslant C\epsilon^2 \tag{7.36}$$

as $\epsilon \downarrow 0$. This contrasts with the fact that u^* and v^* approach zero, in the supremum norm on $\mathbb{R} \times [0, T]$, only at the rate ϵ, as $\epsilon \downarrow 0$.

8. Comments and extensions

The arguments given in §§ 3–7 are capable of dealing with considerably more general equations than the Korteweg–de Vries equation. We eschewed formulating results for more general equations in order to make the argument as transparent as we could. In this section, without going into great detail, we will indicate some of the more or less immediate generalizations of the results already obtained. Of particular concern will be generalizations of interest from the point of view of modelling long waves. It will be shown that a satisfactory theory for both K.–dV. and (1.1) can be formulated in the presence of dissipative and forcing effects.

Both the K.–dV. equation and the model equation (1.1) are 'energy' conserving equations. While this is a very good approximation for long waves even over reasonably long time scales, over dozens of wavelengths dissipation is clearly discernible in some physical systems (e.g. surface water waves, cf. Hammack (1973) and Hammack & Segur (1974)). Thus for certain considerations, it may be desirable to add a dissipative term to the model. This could lead to the following equations, analogous to (1.1) and K.–dV. respectively.

$$u_t + u_x + uu_x - \alpha u_{xx} - u_{xxt} = 0, \tag{8.1}$$

$$u_t + u_x + uu_x - \alpha u_{xx} + u_{xxx} = 0, \tag{8.2}$$

where $\alpha \geqslant 0$. (8.2) has been considered, in generalized form, by Tsutsumi & Mukasa (1971).

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The methods used to treat (1.1) and K.–dV. work just as well for the initial-value problem for (8.1) and (8.2) posed on the entire real axis, and lead to improved existence and continuous dependence results in the case of (8.2).

Consider first the initial-value problem for (8.1). The model may be cast into integral equation form, as with (1.1), obtaining

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\{u(y,\tau) + \tfrac{1}{2}u^2(y,\tau) - \alpha u_x(y,\tau)\}\,dy\,d\tau, \tag{8.3}$$

where g is the initial profile and K is the kernel defined below (3.4). As discussed in the proof of lemma 1, convolution with K maps H^k linearly and continuously to H^{k+1}. Thus

$$\|K*u\|_{k+1} \leqslant C_1\|u\|_k. \tag{8.4}$$

This fact is utilized to show existence of a solution over a small time interval. Suppose $g\in H^k$ for $k \geqslant 1$ (weaker hypotheses suffice at this point, but not subsequently). For $v\in \mathscr{H}^k_T$ let

$$Av(x,t) = g(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\{v + \tfrac{1}{2}v^2 - \alpha v_x\}\,dy\,d\tau. \tag{8.5}$$

Then, $A\colon \mathscr{H}^k_T \to \mathscr{H}^k_T$ is a contraction mapping of a ball about zero in $\mathscr{H}^k_T$ for T sufficiently small. It will follow that A has a fixed point $u\in\mathscr{H}^k_T$ which is then a solution of the integral equation (8.3).

That $A\colon \mathscr{H}^k_T \to \mathscr{H}^k_T$ follows from the mapping properties of K mentioned already and the fact that $g\in H^k$ (cf. proposition 1). The estimates needed to establish the contractive properties of A in a suitable ball are as follows. Let v_1, $v_2\in\mathscr{H}^k_T$, with $\|v_i\|_{\mathscr{H}^k_T} \leqslant R$, $i = 1, 2$. Then if $0 \leqslant t \leqslant T$,

$$\begin{aligned}\|Av_1 - Av_2\|_k &\leqslant t[(1+\alpha C_1)\,\|v_1-v_2\|_k + \tfrac{1}{2}\|v_1^2 - v_2^2\|_k]\\ &\leqslant t[(1+\alpha C_1)\,\|v_1-v_2\|_k + \tfrac{1}{2}C_2(\|v_1\|_k + \|v_2\|_k)\,\|v_1-v_2\|_k]\\ &\leqslant t[1+\alpha C_1 + C_2 R]\,\|v_1-v_2\|_k,\end{aligned}$$

where C_2 depends only on k. Taking the supremum over t in $[0, T]$ yields

$$\|Av_1 - Av_2\|_{\mathscr{H}^k_T} \leqslant T[1+\alpha C_1 + C_2 R]\,\|v_1-v_2\|_{\mathscr{H}^k_T}. \tag{8.6}$$

A second inequality may be derived from (8.6). Let v be again in the ball of radius R about zero in $\mathscr{H}^k_T$. Then since $A(0) = g$,

$$\begin{aligned}\|Av\|_{\mathscr{H}^k_T} &= \|Av - A(0) + g\|_{\mathscr{H}^k_T} \leqslant \|Av - A(0)\|_{\mathscr{H}^k_T} + \|g\|_{\mathscr{H}^k_T}\\ &\leqslant \|g\|_k + T[1+\alpha C_1 + RC_2]\,\|v\|_{\mathscr{H}^k_T}\\ &\leqslant \|g\|_k + T[1+\alpha C_1 + RC_2]\,R.\end{aligned} \tag{8.7}$$

Hence the choice $R = 2\|g\|_k$ and $T = 1/\{2[1+\alpha C_1 + RC_2]\}$ yields the following inequalities, valid for v, v_1, v_2 in $\mathscr{H}^k_T$ with norm less than or equal to R.

$$\left.\begin{aligned}\|Av_1 - Av_2\|_{\mathscr{H}^k_T} &\leqslant \tfrac{1}{2}\|v_1 - v_2\|_{\mathscr{H}^k_T},\\ \|Av\|_{\mathscr{H}^k_T} &\leqslant R.\end{aligned}\right\} \tag{8.8}$$

Thus A is seen to be a contractive mapping of the closed ball of radius R about zero in $\mathscr{H}^k_T$ and hence there is a $u\in\mathscr{H}^k_T$, with norm at most R such that

$$Au = u. \tag{8.9}$$

Now standard bootstrap arguments (cf. Benjamin *et al.* 1972, lemma 2) show that a fixed point u is in fact a solution of the differential equation over the time interval $[0, T]$.

These arguments are extended to yield a global solution of the initial-value problem by deriving *a priori* bounds for solutions of the initial-value problem for (8.1). The derivation follows arguments similar to those given earlier and we merely anticipate the result that if $g \in H^1 \cap C_b^2$ then so is $u(x, t)$ for each $t \in [0, T]$ and further

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} [u^2(x,t) + u_x^2(x,t)]\,\mathrm{d}x + 2\alpha \int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x = 0. \tag{8.10}$$

From this it follows that the H^1 norm of $u(x, t)$ is decreasing with increasing time. Hence the proof of existence in the small can be iterated to yield a global solution to the problem just as in Benjamin *et al.* (1972, pp. 61–62). The extension to higher order Sobolev spaces can now proceed from the integral equation (8.5) just as in the proof of lemma 1 in § 2. Here is the precise result.

Proposition 13. Let $g \in H^m$ where $m \geqslant 2$. Then there is a unique solution u in $\mathscr{H}_\infty^m$, with initial value g, to the equation (8.1). Furthermore, $\partial_t^k u \in \mathscr{H}_T^m$ for all $k \geqslant 0$ and finite $T > 0$.

The only point that requires comment is the claim that u is bounded in H^m uniformly in time. This follows from *a priori* estimates which will be outlined below in the attack on the equation (8.2) (cf. (8.16), (8.21) and (8.22) with $\epsilon = 1$).

To tackle the equation (8.2) use is made of the theory for the initial-value problem for (8.1). First a shift to coordinates moving at speed one gives a slightly simpler initial-value problem.

$$\left.\begin{aligned} u_t + uu_x - \alpha u_{xx} + u_{xxx} &= 0, \\ u(x,0) &= g(x). \end{aligned}\right\} \tag{8.11}$$

This initial-value problem is regularized as before by addition of a term $-\epsilon u_{xxt}$. Thus consideration is given to the problem

$$\left.\begin{aligned} u_t + uu_x - \alpha u_{xx} + u_{xxx} - \epsilon u_{xxt} &= 0, \\ u(x,0) &= g(x). \end{aligned}\right\} \tag{8.12}$$

Letting v be defined from u by the change of variables (3.2), there appears the following initial-value problem for v

$$\left.\begin{aligned} v_t + v_x + vv_x - \alpha\epsilon^{-\frac{1}{2}} v_{xx} - v_{xxt} &= 0, \\ v(x,0) &= \epsilon g(\epsilon^{\frac{1}{2}} x). \end{aligned}\right\} \tag{8.13}$$

For fixed $\epsilon > 0$, proposition 13 assures existence of smooth solutions of (8.13) obtain for given data $g \in H^m$, $m \geqslant 2$. Hence exactly as in lemma 2, smooth solutions to the regularized initial-value problem (8.12) obtain without further difficulty.

Derivation of ϵ-independent bounds for the solutions of (8.12) is actually much easier with the dissipative term included. However, if α-independent bounds are desired one must proceed as before in § 4. But for a fixed level of dissipation, the following simpler arguments are available. Suppose $g \in H^\infty$ so that u is a C^∞ function of both variables all of whose derivatives are in L_2. Then upon multiplying the regularized equation by $u_{(2k)}$ for $k = 0, 1, 2, \ldots$, and integrating over $\mathbb{R}$ and $[0, t]$, and after appropriate integrations by parts, there appears

$$\int_{-\infty}^{\infty} (u_{(k)}^2 + \epsilon u_{(k+1)}^2)\,\mathrm{d}x + 2\alpha \int_0^t \int_{-\infty}^{\infty} u_{(k+1)}^2\,\mathrm{d}x\,\mathrm{d}\tau = \int_{-\infty}^{\infty} (g_{(k)}^2 + \epsilon g_{(k+1)}^2)\,\mathrm{d}x - \int_0^t \int_{-\infty}^{\infty} (u^2)_{(k)} u_{(k+1)}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{8.14}$$

For $k = 0$ (8.14) is the analogue of (8.10), namely

$$\int_{-\infty}^{\infty} (u^2 + \epsilon u_x^2)\,\mathrm{d}x + 2\alpha \int_0^t \int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x = \int_{-\infty}^{\infty} (g^2 + \epsilon g_x^2)\,\mathrm{d}x. \tag{8.15}$$

From this, independently of ϵ in $(0, 1]$ and of $t \geqslant 0$,

$$\|u\| \leqslant C_0, \quad \int_0^t \int_{-\infty}^{\infty} u_x^2 \, dx \, d\tau \leqslant C_0, \tag{8.16}$$

where $C_0 = C_0(\|g\| + \epsilon\|g'\|)$ which can be taken to be positive without loss of generality (the case $g \equiv 0$ being trivial in all aspects). Now let $k = 1$ in the master relation (8.14). Then

$$\left.\begin{aligned} I_1 &= \int_{-\infty}^{\infty} (u_x^2 + \epsilon u_{xx}^2) \, dx + 2\alpha \int_0^t \int_{-\infty}^{\infty} u_{xx}^2 \, dx \, d\tau \\ &= \int_{-\infty}^{\infty} (g_x^2 + \epsilon g_{xx}^2) \, dx - 2 \int_0^t \int_{-\infty}^{\infty} u u_x u_{xx} \, dx \, d\tau \\ &\leqslant C + 2 \int_0^t \|u\| \, \|u_{xx}\| \, \|u_x\|_\infty \, d\tau \\ &\leqslant C + 2C_0 \int_0^t \|u_{xx}\| \, \|u_x\|_\infty \, d\tau \\ &\leqslant C + 2C_0 \left(\int_0^t \|u_x\|_\infty^2 \, d\tau \int_0^t \|u_{xx}\|^2 \, d\tau \right)^{\frac{1}{2}}, \end{aligned}\right\} \tag{8.17}$$

making use of (8.16) and the Schwarz inequality applied once in each variable. Now use the elementary inequality, valid for all $\gamma > 0$,

$$2AB \leqslant \gamma A^2 + (1/\gamma) B^2. \tag{8.18}$$

Then the estimate (8.17) continues as follows.

$$I_1 \leqslant C + C_0 \left(\frac{2C_0}{\alpha} \int_0^t \|u_x\|_\infty^2 \, d\tau + \frac{\alpha}{2C_0} \int_0^t \|u_{xx}\|^2 \, d\tau \right). \tag{8.19}$$

Now from (4.5), $\|u_x\|_\infty^2 \leqslant \|u_x\| \, \|u_{xx}\|$. Thus applying the Schwarz inequality and (8.18) again, with $\gamma = 4C_0^2/\alpha^2$

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} (u_x^2 + \epsilon u_{xx}^2) \, dx + 2\alpha \int_0^t \int_{-\infty}^{\infty} u_{xx}^2 \, dx \, d\tau \\ &\leqslant C + C_0 \left(\frac{4C_0^3}{\alpha^3} \int_0^t \|u_x\|^2 \, d\tau + \frac{\alpha}{C_0} \int_0^t \|u_{xx}\|^2 \, d\tau \right) \\ &\leqslant C + \alpha \int_0^t \int_{-\infty}^{\infty} u_{xx}^2 \, dx \, d\tau + 4\frac{C_0^4}{\alpha^3} \int_0^t \int_{-\infty}^{\infty} u_x^2 \, dx \, d\tau \\ &\leqslant C + 4\frac{C_0^5}{\alpha^3} + \alpha \int_0^t \int_{-\infty}^{\infty} u_{xx}^2 \, dx \, d\tau. \end{aligned}$$

Thus in sum,

$$\int_{-\infty}^{\infty} (u_x^2 + \epsilon u_{xx}^2) \, dx + \alpha \int_0^t \int_{-\infty}^{\infty} u_{xx}^2 \, dx \, d\tau \leqslant C + 4\frac{C_0^5}{\alpha^3} = C_1, \tag{8.20}$$

where $C_1 = C_1(\|g\|_1 + \epsilon\|g''\|)$. It follows now that independently of ϵ in $(0, 1]$ and of $t \geqslant 0$,

$$\int_{-\infty}^{\infty} u_x^2 \, dx \leqslant C_1, \quad \int_0^t \int_{-\infty}^{\infty} u_{xx}^2 \, dx \, d\tau \leqslant C_1'. \tag{8.21}$$

One may proceed inductively to deduce the bounds

$$\int_{-\infty}^{\infty} u_{(k)}^2 \, dx \leqslant C_k, \quad \int_0^t \int_{-\infty}^{\infty} u_{(k+1)}^2 \, dx \, d\tau \leqslant C_k, \tag{8.22}$$

for all $k \geqslant 0$ where $C_k = C_k(\|g\|_k + \epsilon\|g_{(k+1)}\|)$ is independent of ϵ in $(0, 1]$ and $t \geqslant 0$. Note that these bounds are not restricted to finite time intervals as were the corresponding bounds in proposition 4 for the regularized K.–dV. equation with no dissipation.

Passage to the limit may now be effected by the methods of § 5 or of appendix A. We may suitably pass over the details, which parallel those already set down except for the additional linear term $-\alpha u_{xx}$, which causes no difficulty. Uniqueness is established just as before. We sum up the situation as follows.

THEOREM 7. Let $g \in H^m$, where $m \geqslant 2$. Then there is a unique solution u in $\mathscr{X}_{m,\infty}$ to the dissipative K.–dV. equation (8.2) with initial value g. u depends continuously in $\mathscr{X}_{m,\infty}$ on g in H^m.

Further 'interior' regularity results may be derived for $t > 0$ by using Fourier analysis and general regularity results for the non-homogeneous heat equation. This point will not be dealt with here however.

For many physical systems there can be direct external forcing of the waves. Mathematically, this leads to the nonhomogeneous model problems

$$u_t + u_x + uu_x - u_{xxt} = f(x, t), \tag{8.23}$$

and

$$u_t + u_x + uu_x + u_{xxx} = f(x, t). \tag{8.24}$$

Of course dissipative effects can be combined in these models with the forcing effects if desired. As demonstrated in the analysis leading to theorem 7, dissipation makes matters better in general.

An advantage of (8.23) over (8.24) is that weaker assumptions appear to be needed on the forcing function in order to insure a given smoothness of solutions. It would be expected that f need be only as regular as the least regular term on the left hand side of either (8.23) or (8.24). This obtains for (8.23), but we have only been able to prove a weaker result for (8.24). Examination of the linear problem obtained from (8.24) by dropping the nonlinear term suggests, but does not prove, that perhaps the stronger assumptions made below on f in order to treat (8.24) are just in the nature of things. In any case, here are statements of the precise results we can establish.

PROPOSITION 14. Let $g \in H^s$, $s \geqslant 1$ and $f \in \mathscr{H}_T^{s-1}$ for some $T > 0$. Then there exists a unique solution u in $\mathscr{H}_T^s$ to (8.23) which takes the initial value g. Furthermore, $\partial_t u \in \mathscr{H}_T^{s+1}$ and if $\partial_t^j f \in \mathscr{H}_T^l$ for some non-negative $l \leqslant s-1$ for all j with $1 \leqslant j \leqslant m$, then $\partial_t^k u \in \mathscr{H}_T^{l+2}$ for $2 \leqslant k \leqslant m+1$. The solution u depends continuously in $\mathscr{H}_T^s$ on g in H^s and f in $\mathscr{H}_T^{s-1}$.

PROPOSITION 15. Let $g \in H^s$, $s \geqslant 3$, and let $f \in \mathscr{H}_T^s$ and also $f_t \in \mathscr{H}_T$. Then there is a unique solution u in $\mathscr{H}_T^s$ to (8.24) with initial value g. Further, if $f \in \mathscr{X}_{s,T}$ (see (2.4)) then $u \in \mathscr{X}_{s,T}$. The solution u depends continuously in $\mathscr{H}_T^s$ on g in H^s and f in $\mathscr{H}_T^s \cap C^1(0, T; H^0)$ (resp. in $\mathscr{X}_{s,T}$ on g in H^s and f in $\mathscr{X}_{s,T}$).

Proposition 15 may be viewed as an improvement of the results for (8.24) announced by Kametaka (1969).

We content ourselves with a few remarks concerning the proofs of the last two propositions. Proposition 14 requires only a mild extension of the theory developed for (8.23) by Benjamin *et al.* (1972, theorem 2). The extension is made along the lines of lemma 1 in § 2, by using an associated integral equation.

The proof of proposition 15 is essentially the same as the proof given in §§ 4, 5 and 6 of theorems 1 and 4. We note a few of the more interesting changes.

Proof of existence for a regularized version of (8.24), obtained by passing to moving coordinates to dispense with the term u_x and then adding the term $-\epsilon u_{xxt}$, is effected by the change of variables (3.2) and the result of proposition 14. The derivation of *a priori* bounds undertaken in propositions 2, 3 and 4 proceeds as before except the identities change slightly and the bounds necessarily depend on the forcing function f. For example, the identities corresponding to (4.2) and (4.3) in proposition 2 now take the following form

$$\int_{-\infty}^{\infty} (u^2+\epsilon u_x^2)\,dx = \int_{-\infty}^{\infty} (g^2+\epsilon g_x^2)\,dx + 2\int_0^t\int_{-\infty}^{\infty} uf\,dx\,d\tau, \tag{8.25}$$

and

$$\int_{-\infty}^{\infty} (u_x^3-\tfrac{1}{3}u^3)\,dx = \int_{-\infty}^{\infty} (g_x^2-\tfrac{1}{3}g^3)\,dx + \int_0^t\int_{-\infty}^{\infty} (2u_x f_x - u^2 f - 2\epsilon u_x f_t)\,dx\,d\tau$$
$$+2\epsilon\int_{-\infty}^{\infty} [f(x,t)\,u_x(x,t) - f(x,0)\,u_x(x,0)]\,dx. \tag{8.26}$$

From these two identities it follows much as in proposition 2 that u is bounded in $\mathscr{H}_T^1$ independently of ϵ in $(0, 1]$. Similarly the analogue of (4.9) in proposition 3 is

$$V(t) = V(0) - \epsilon\int_0^t\int_{-\infty}^{\infty} (3u_t u_{xx}^2 + 3u^2 u_x u_{xt} + 6u_x u_{xx} u_{xt})\,dx\,d\tau$$
$$+\int_0^t\int_{-\infty}^{\infty} (u^3 f + 3u_x^2 f - 6u_x^2 f - 6uu_x f_x + \tfrac{18}{5}u_{xx} f_{xx})\,dx\,d\tau, \tag{8.27}$$

where as defined below (4.8)

$$V(t) = \int_{-\infty}^{\infty} [(\tfrac{9}{5}-3\epsilon u)\,u_{xx}^2 - 3uu_x^2 + \tfrac{1}{4}u^4 + \epsilon u_{xxx}^2]\,dx. \tag{8.28}$$

The assumptions on f coupled with the already derived H^1 bound on u suffices to bound the second integral in (8.27) on $[0, T]$ by a constant independent of ϵ in $(0, \epsilon_0]$ where ϵ_0 is to satisfy (4.10). Thus (4.17) is recovered where the constant C now depends on f as well as $\|g\|_3$. Now differentiating the regularized equation with respect to t, multiplying by u_t and integrating over $\mathbb{R}$ and over $[0, t]$ yields the analogue of (4.19).

$$B^2(t) \leqslant B^2(0) + \int_0^t [\|u_x\|_\infty B^2(\tau) + 2\|f_t\|\,B(\tau)]\,d\tau, \tag{8.29}$$

where as in (4.18)

$$B^2(t) = \int_{-\infty}^{\infty} (u_t^2 + \epsilon u_{xt}^2)\,dx. \tag{8.30}$$

$B^2(0)$ is estimated as before in lemma 4 and then (4.17) and (8.29) are played together to obtain that for any $T > 0$, u is bounded in $\mathscr{H}_T^2$ independently of sufficiently small ϵ. Further bounds analogous to those obtained in proposition 4 follow by arguments which differ little from those presented in the proof of proposition 4.

At this juncture, the arguments of § 5 may be imitated to conclude existence of a smooth solution as advertized in proposition 15. It is required to regularize the forcing function f in both x and t, but this presents no difficulty. Continuous dependence results now follow readily.

Other generalizations of the basic K.–dV. model or the model (1.1) can be handled by the same methods. For example, much more general nonlinear terms can be handled and more general dispersion relations can be accommodated. It is not intended to explore these possibilities here – the problem of more general dispersion in particular, while interesting for both modelling and mathematical reasons, would lead rather far afield.

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Appendix A. An alternative proof of the convergence of solutions of the regularized problem to solutions of K.–dV.

Here we consider some mathematical refinements of the theory presented in §§ 3–5 and examine an alternative approach to the K.–dV. initial-value problem. The methods call upon additional results from functional analysis, but allow the range of admissible initial data to be extended slightly.

Theorem 8. Let $g \in H^k$, where $k \geq 1$ is an integer. Then the K.–dV. equation has a solution in $L^\infty(0, \infty; H^k)$ with initial value g. If $k \geq 2$, then the solution is unique in this function class.

Remarks. The present notation for function spaces follows that of Lions & Magenes (1968, vol. 1, ch. 1). If $k < 3$, the fact that u is a solution of K.–dV. may have to be interpreted in the distribution sense. Note that from the differential equation, $u_t \in L^\infty(0, \infty; H^{k-3})$, so that u is in $C(0, \infty; H^{k-\frac{3}{2}})$ (Lions & Magenes 1968, vol. 1, p. 23) and therefore the initial condition is satisfied in a meaningful manner. If $k \geq 3$, all the derivatives in question in the K.–dV. equation exist almost everywhere and the equation is satisfied pointwise almost everywhere. Of course if $k \geq 3$, the solution must be the same solution obtained in theorem 1, by the uniqueness result, and hence is a classical solution for $k > 3$.

The result of theorem 8 is an improvement on theorem 1 for $k < 3$. For the particular case $k = 2$, the present results, expounded in theorem 8, will be improved subsequently in this appendix.

Proof. The proof is made in two steps. First, existence of smooth solutions of K.–dV. corresponding to smooth initial data is established. Then a limit is taken through smooth solutions of K.–dV. to infer existence of solutions corresponding to data in H^k.

Accordingly, let $g \in H^\infty$ and let u_ϵ be the smooth (it is C^∞ with all its derivatives in L_2, by the corollary to lemma 2) solution to the regularized initial-value problem. Then proposition 4 assures that for any $T > 0$, $s \geq 0$ and independently of ϵ in $(0, \epsilon_0]$ u_ϵ is bounded in $L^\infty(0, T; H^s)$. It follows from the regularized differential equation that $\partial_t u_\epsilon$ is bounded in $L^\infty(0, T; H^s)$ for all $T > 0$, $s \geq 0$ and independent of ϵ in $(0, \epsilon_0)$. Thus a diagonalization argument allows the conclusion that there is a sequence $\epsilon_n \downarrow 0$ and a u which is $L^\infty(0, T; H^s)$, for all $s \geq 0$, such that

$$\left.\begin{aligned} u_n = u_{\epsilon_n} &\to u \quad \text{weak-star in} \quad L^\infty(0, T; H^s), \\ \partial_t u_n &\to u_t \quad \text{weak-star in} \quad L^\infty(0, T; H^s), \end{aligned}\right\} \tag{A 1}$$

for all $T > 0$ and $s \geq 0$. Note that from the first line of (A 1), $\partial_t u_n \to \partial_t u$ in $\mathscr{D}'(0, T; H^s)$, hence the weak-star limit in the second line of (A 1) necessarily converges to u_t. Note also that here and below, the taking of subsequences turns out not to be necessary for $k \geq 2$. For the uniqueness result will allow the conclusion that any subsequence of $\{u_\epsilon\}$ has a further subsequence which converges to the unique solution u of the equation. It follows that $\{u_\epsilon\}$ converges to u as $\epsilon \downarrow 0$.

Lemma 7. Suppose $u_n \to u$ weak-star in $L^\infty(0, T; H^s)$ where $s > \frac{1}{2}$ and $\partial_t u_n \to \partial_t u$ weak-star in $L^2(0, T; H^r)$ for some real r. Then there exists a subsequence $\{u_l\}$ of $\{u_n\}$ such that $u_l \to u$ pointwise almost everywhere in $[0, T] \times \mathbb{R}$ and $u_l \partial_x u_l \to uu_x$ in $\mathscr{D}'([0, T] \times \mathbb{R})$ in the usual sense of distributions (uu_x is interpreted as $\frac{1}{2}\partial_x(u^2)$ in case $s < 1$ and similarly for $u_l \partial_x u_l$).

Proof. Let $\{\Omega_m\}$ be an increasing sequence of bounded open intervals in $\mathbb{R}$ with $\bigcup_m \Omega_m = \mathbb{R}$, and let $H^s(\Omega_m)$ be the Sobolev space of order s, defined for example in (Lions & Magenes 1968, vol. I, ch. 1, § 9). Since $\{u_n\}$ must be bounded in $L^\infty(0, T; H^s)$, it follows that $\{u_n\}$ is bounded in $L^2(0, T; H^s(\Omega_m))$ for all m. Further, for all m, $\{u_n'\}$ is bounded in $L^2(0, T; H^r(\Omega_m))$ where, without loss of generality, $r \leqslant 0$. (Here and below, $u_n' = \partial_t u_n$.)

As Ω_m is bounded and $s > 0$, $H^s(\Omega_m)$ is compactly imbedded in $L_2(\Omega_m)$. An application of Lions 1969, ch. 1, théorème 5.1 shows that $L^2(0, T; H^s(\Omega_m))$ is compactly imbedded in $L^2(0, T; L_2(\Omega_m)) = L_2(Q_m)$, where $Q_m = [0, T] \times \Omega_m$, for each m. By using these compact imbeddings and an additional diagonalization argument, it is concluded that there is a subsequence $\{u_j\}$ of $\{u_n\}$ such that for all m,

$$u_j \to u \quad \text{strongly in} \quad L^2(Q_m). \tag{A 2}$$

Hence a further diagonalization argument leads to a further subsequence $\{u_l\}$ such that (A 2) holds and

$$u_l \to u, \quad \text{pointwise almost everywhere in} \quad Q = [0, T] \times \mathbb{R}. \tag{A 3}$$

To establish the claim, it will suffice to show that $u_l^2 \to u^2$ in $\mathscr{D}'(Q)$ since differentiation is a continuous operation in $\mathscr{D}'(Q)$. Hence it must be demonstrated that

$$\int_0^T \int_{-\infty}^{\infty} (u_l^2 - u^2)\, \varphi \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as} \quad l \to \infty, \tag{A 4}$$

for any test function φ in $\mathscr{D}(Q)$. Since $s > \frac{1}{2}$, $H^s \hookrightarrow L_\infty$ and so $L^\infty(0, T; H^s) \hookrightarrow L^\infty(0, T; L_\infty) = L^\infty(Q)$ Therefore $\{u_l\}$ is bounded in $L^\infty(0, T; H^s)$ implies $\{u_l\}$ is bounded in $L^\infty(Q)$, say

$$\|u_l\|_{L^\infty(Q)} \leqslant M \quad \text{for all} \quad l. \tag{A 5}$$

Then $|(u_l^2 - u^2)\,\phi| \leqslant 2M^2\varphi$ and from (A 3) $(u_l^2 - u^2)\,\varphi \to 0$ pointwise almost everywhere in Q. Lebesgue's dominated convergence theorem now establishes (A 4) and completes the proof of the lemma.

The lemma is applied to the sequence in (A 1) to obtain a subsequence $\{u_l\}$ such that for all $T > 0$ and $s \geqslant 1$,

$$\left.\begin{aligned} u_l &\to u && \text{weak-star in} \quad L^\infty(0, T; H^s), \\ u_l' &\to u' && \text{weak-star in} \quad L^\infty(0, T; H^s), \\ u_l u_{lx} &\to uu_x && \text{in} \quad \mathscr{D}'(Q). \end{aligned}\right\} \tag{A 6}$$

It follows immediately that at least in the sense of distributions $\{u_l\}$ converges to a solution u of the K.–dV. equation. Since all the u_l have the initial value g, a simple argument (Lions 1969, ch. 1, p. 14) shows that u does as well and therefore u is a solution of the K.–dV. initial-value problem which lies in $L^\infty(0, T; H^s)$, for all $T > 0$ and $s \geqslant 0$. It then follows inductively from the differential equation that u is a C^∞ function of (x, t) all of whose partial derivatives belong to L_2 for each fixed $t \geqslant 0$.

For such smooth solutions of K.–dV. there is at hand the whole range of polynomial invariants $I_0(u), I_1(u), \ldots$ in (5.36). Proposition 6 thus shows that u is bounded, with all its partial derivatives, in H^s *uniformly* for $t \geqslant 0$.

Now we pass to the second stage of the proof, where it is supposed g is in H^k for some $k \geqslant 1$. Let $\{g_n\}$ be a sequence in H^∞ such that $g_n \to g$ strongly in H^k (cf. proposition 5 for an explicit construction of such smooth approximations). Let u_n be the smooth solution of the K.–dV. equation with initial data g_n. It follows from the strong convergence of $\{g_n\}$ to g in H^k that

$$I_j(g_n) \to I_j(g) \quad \text{for} \quad 0 \leqslant j \leqslant k, \tag{A 7}$$

as $n \to \infty$. Furthermore, from theorem 2, for any $t \geqslant 0$,

$$I_j(u_n) = I_j(g_n). \tag{A 8}$$

It then follows that, independent of n and of $t \geqslant 0$, $I_j(u_n)$ is bounded for $0 \leqslant j \leqslant k$. By using the general form of I_j given in (5.36) and proceeding inductively from $j = 0$ as indicated near (5.37) it may be concluded that $\|u_n\|_k$ is bounded, independent of n and of $t \geqslant 0$. Thus $\{u_n\}$ is a bounded sequence in $L^\infty(0, \infty; H^k)$ and hence as in (A 1) and lemma 7, there is a subsequence $\{u_m\}$ of $\{u_n\}$ and a u in $L^\infty(0, \infty; H^k)$ such that

$$\left.\begin{aligned} u_m &\to u \quad \text{weak-star in} \quad L^\infty(0, \infty; H^k), \\ u_m' &\to u' \quad \text{weak-star in} \quad L^\infty(0, \infty; H^{k-3}), \\ u_m u_{mx} &\to u u_x \quad \text{in} \quad \mathscr{D}'([0, \infty) \times \mathbb{R}). \end{aligned}\right\} \tag{A 9}$$

It follows that u is a solution of the K.–dV. equation and moreover $u(x, 0) = g(x)$ as one sees by deducing from (A 9) that $u_n(x, 0) \to u(x, 0)$ weakly in H^{k-3} (cf. again Lions 1969, ch. 1, p. 14) and then recalling that $u_n(x, 0) = g_n(x) \to g(x)$ strongly in H^k.

Uniqueness for the case $k \geqslant 2$ follows the standard lines given already in the proof of theorem 1. For the case $k = 2$, a slight amount of care is needed to interpret the argument, but we may safely refer to the remarks of Temam (1969, pp. 170–171), made for the periodic problem, which carry over without change to the present situation. The proof of the theorem is now complete.

Corollary 1. The solution u guaranteed by theorem 8 is, after possible modification on a null set of $t \in \mathbb{R}$, weakly continuous from $\mathbb{R}^+$ to H^k.

Proof. This follows by first interpolating to see that u is in $C(0, \infty; H^{k-\frac{3}{2}})$ and then applying Lions & Magenes (1968, lemma 8.1, ch. 3, p. 297) and the fact $u \in L^\infty(0, \infty; H^k)$ to infer the stated weak continuity.

To obtain the strong continuity in time of the above solution, as in theorem 1, one could use the first stage of the proof of theorem 8 to establish existence of smooth solutions corresponding to smooth data, and then use an argument analogous to that given in proposition 5 (e.g. use (5.10–5.11), etc., with $\epsilon = \delta = 0$) to show that solutions u_n, corresponding to smooth data g_n which approximate g in H^k appropriately for $k \geqslant 2$, are Cauchy in $C(0, T; H^k)$ for all $T > 0$. It follows that $u_n \to u$ strongly in $C(0, T; H^k)$ for all $T > 0$ and thus u is strongly continuous (it is the same u as obtained in theorem 8 by the uniqueness result).

It is of some interest to note that if the argument for existence is made as last outlined, theorem 1 may be improved by taking account in this way of initial data in H^2. We thus obtain a slightly strengthened version of theorem 1 which is stated formally as another corollary.

Corollary 2. Let $g \in H^k$ for an integer $k \geqslant 2$. Then there exists a unique solution u in $C_b(0, \infty; H^k)$ to the K.–dV. equation with initial value g.

Proof. The improvement comes about from the superior H^2 estimates one obtains by first passing to the limit of the regularized equation as $\epsilon \downarrow 0$ for H^∞ data and then using the K.–dV. invariants for *a priori* bounds rather than the direct derivation of H^2 bounds given for the regularized equation in proposition 3.

Specifically, let $g \in H^2$ and let g_ϵ be defined as in (5.1) and let u_ϵ be the solution of the K.–dV. initial-value problem for the H^∞ initial data g_ϵ, guaranteed by theorem 1 or 8. Then as in the argument proving theorem 8, the first three polynomial invariants for smooth solutions of K.–dV. can be used to deduce that independently of ϵ in $(0, 1]$ and of $t \geqslant 0$,

$$\|u_\epsilon\|_2 \leqslant M. \tag{A 10}$$

From lemma 5 and further polynomial invariants,

$$\|u_\epsilon\|_{2+k} \leqslant M\epsilon^{-\frac{1}{6}k} \quad \text{as} \quad \epsilon \downarrow 0, \tag{A 11}$$

for all $k \geqslant 0$. Let $T > 0$ be fixed. Then $\{u_\epsilon\}$ is Cauchy in $C(0, T; H^2)$. To see this, let $u = u_\epsilon$ and $v = u_\delta$ where $\epsilon \geqslant \delta > 0$ say, and let $w = u - v$. Then just as in (5.11), for any $j \geqslant 0$,

$$\int_{-\infty}^{\infty} w_{(j)}^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} h_{(j)}^2 \,\mathrm{d}x - 2\int_0^t \int_{-\infty}^{\infty} (uw + \tfrac{1}{2}w^2)_{(j+1)} w_{(j)} \,\mathrm{d}x \,\mathrm{d}\tau, \tag{A 12}$$

where $h(x) = u(x, 0) - v(x, 0) = g_\epsilon(x) - g_\delta(x)$. From lemma 5 it appears that

$$\|h\|_{2-k} = o(\epsilon^{\frac{1}{6}k}) \quad \text{for} \quad k = 0, 1, 2. \tag{A 13}$$

Applying (A 10) to (A 12) with $j = 0$ leads to

$$\int_{-\infty}^{\infty} w^2 \,\mathrm{d}x \leqslant C_0 \int_0^t \int_{-\infty}^{\infty} w^2 \,\mathrm{d}x + \int_{-\infty}^{\infty} h^2 \,\mathrm{d}x, \tag{A 14}$$

where $C_0 = C_0(\|u\|_2, \|v\|_2)$ is bounded independent of ϵ in $(0, 1]$. From this it is deduced

$$\|w\| \leqslant \|h\| \,\mathrm{e}^{\frac{1}{2}C_0 T} = o(\epsilon^{\frac{1}{3}}), \tag{A 15}$$

uniformly for t in $[0, T]$ as $\epsilon \downarrow 0$. (A 12) with $j = 1$ can be expressed as

$$V_1(t)^2 = \int_{-\infty}^{\infty} w_x^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} h_x^2 \,\mathrm{d}x - 2\int_0^t \int_{-\infty}^{\infty} [(\tfrac{1}{2}w_x + \tfrac{3}{2}u_x)\, w_x^2 + u_{xx} w w_x] \,\mathrm{d}x \,\mathrm{d}\tau.$$

This generates the inequality

$$\begin{aligned} V_1(t)^2 &\leqslant V_1(0)^2 + C_1 \int_0^t V_1(\tau)^2 \,\mathrm{d}\tau + 2\int_0^t \|u_{xx}\|_\infty \|w\| \, \|w_x\| \,\mathrm{d}\tau \\ &\leqslant V_1(0)^2 + C_1 \int_0^t V_1(\tau)^2 \,\mathrm{d}\tau + 2 \sup_{0 \leqslant t \leqslant T} \|u\|_3 \sup_{0 \leqslant t \leqslant T} \|w\| \int_0^t V_1(\tau) \,\mathrm{d}\tau. \end{aligned}$$

Here $C_1 = C_1(\|u\|_2, \|v\|_2)$ is bounded independent of ϵ in $(0, 1]$. Now $\sup \|u\|_3 = O(\epsilon^{-\frac{1}{6}})$ whereas $\sup \|w\| = o(\epsilon^{\frac{1}{3}})$ as $\epsilon \downarrow 0$. Hence, uniformly on $[0, T]$

$$V_1(t)^2 \leqslant V_1(0)^2 + C_1 \int_0^t V_1(\tau)^2 \,\mathrm{d}\tau + o(\epsilon^{\frac{1}{6}}) \int_0^t V_1(\tau) \,\mathrm{d}\tau. \tag{A 16}$$

(A 16) then implies immediately the inequality

$$\|w_x\| = V_1(t) \leqslant \|h_x\| \,\mathrm{e}^{\frac{1}{2}C_1 T} + o(\epsilon^{\frac{1}{6}})\, C_1^{-1}(\mathrm{e}^{\frac{1}{2}C_1 T} - 1), \tag{A 17}$$

valid for t in $[0, T]$. (A 13) now comes to our aid and yields

$$\|w_x\| = o(\epsilon^{\frac{1}{6}}) \quad \text{as} \quad \epsilon \downarrow 0, \tag{A 18}$$

uniformly on $[0, T]$. Finally for $j = 2$ in (A 12), there appears, as in (5.13–5.14),

$$V_2(t)^2 = \int_{-\infty}^{\infty} w_{xx}^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} h_{xx}^2 \,\mathrm{d}x - 2\int_0^t \int_{-\infty}^{\infty} [\tfrac{5}{2}(u_x + w_x)\, w_{xx}^2 + 3u_{xx} w_x w_{xx} + u_{xxx} w w_{xx}] \,\mathrm{d}x \,\mathrm{d}\tau.$$

There follows

$$V_2(t)^2 \leqslant V_2(0)^2 + C_2\int_0^t V_2(\tau)^2\,\mathrm{d}\tau + 6\int_0^t \|u_{xx}\|_\infty\,\|w_x\|\,\|w_{xx}\|\,\mathrm{d}\tau + 2\int_0^t \|u_{xxx}\|_\infty\,\|w\|\,\|w_{xx}\|\,\mathrm{d}\tau.$$

where C_2 is bounded independent of ϵ in $(0, 1]$. But from (A 11), (A 15) and (A 18),

$$\sup_{0\leqslant t\leqslant T}\|u_{xx}\|_\infty\,\|w_x\| \leqslant \sup_{0\leqslant t\leqslant T}\|u\|_3\,\|w_x\| = O(\epsilon^{-\frac{1}{6}})\,o(\epsilon^{\frac{1}{6}}) = o(1),$$

as $\epsilon\downarrow 0$. Similarly, $$\sup_{0\leqslant t\leqslant T}\|u_{xxx}\|_\infty\,\|w\| = O(\epsilon^{-\frac{1}{3}})\,o(\epsilon^{\frac{1}{3}}) = o(1),$$

as $\epsilon\downarrow 0$. Hence uniformly on $[0, T]$,

$$V_2(t)^2 = V_2(0)^2 + C_2\int_0^t V_2(\tau)^2\,\mathrm{d}\tau + o(1)\int_0^t V_2(\tau)\,\mathrm{d}\tau,$$

from which we have $$\|w_{xx}\| = V_2(t) \leqslant \|h_{xx}\|\,\mathrm{e}^{\frac{1}{2}C_2T} + o(1)\,C_2^{-1}(\mathrm{e}^{\frac{1}{2}C_2T}-1).$$

Now (A 13) gives $$\|w_{xx}\| = o(1) \quad \text{as} \quad \epsilon\downarrow 0, \tag{A 19}$$
uniformly for t in $[0, T]$.

Then (A 15–A 18, A 19) taken in conjunction yield the desired property of $\{u_\epsilon\}$ and it is concluded that $u_\epsilon\to u$ in $C(0, T; H^2)$. The argument of theorem 2 then assures that the first three invariants, $I_0(u)$, $I_1(u)$ and $I_2(u)$ do not vary with time. It is then just an application of proposition 6 to conclude $u\in C_b(0,\infty; H^2)$. The proof of the corollary is complete.

Continuous dependence results for the H^2 case are derived exactly as in theorem 4, and we may suitably pass over these details. Further existence results can be derived for initial data g in H^s where $s > \frac{1}{2}$ is not necessarily an integer. Results in this direction have been given by Saut (1974) making use of the nonlinear interpolation theory of Tartar. Additional results will be given in a forthcoming publication in which it will be shown that if $g\in H^s$ where $s\geqslant 2$, then there corresponds a unique solution of K.–dV. which lies in the function class $C(0, T; H^s)$ for any finite $T > 0$. The proof relies on the present theory and an extension of Tartar's theory.

Appendix B. The periodic initial-value problem

An entirely parallel development can be given for the periodic initial-value problem for the K.–dV. equation (again expressed in moving coordinates)

$$u_t + uu_x + u_{xxx} = 0, \quad u(x, 0) = g(x), \tag{B 1}$$

where g is a given periodic function with period L, say. It is not intended to present details of this development, which differ only slightly from the details of the theory for the pure initial-value problem already considered. However, it may be worth while indicating the outcome of the analysis and comparing it with the previously obtained results for the periodic initial-value problem, outlined already in the introduction. In contrast to the pure initial-value problem, where previous results had lost some control of the spatial derivatives in obtaining a solution, solutions u in $L^\infty(0, T; H^k_{\mathrm{loc}})$ corresponding to periodic H^k_{loc} initial data have been shown to exist, where $k\geqslant 1$ and T is positive and fixed, but otherwise arbitrary. We shall be able to strengthen this conclusion, inferring the solutions are in $C_b(0,\infty; H^k_{\mathrm{loc}})$ for periodic H^k_{loc} initial data, $k\geqslant 2$. The analysis begins with a new periodic result for the model equation (1.1), preceding which the spatially periodic versions of the function spaces that have figured thus far are defined.

By $H^k_{\mathrm{loc}} = H^k_{\mathrm{loc}}(\mathbb{R})$ is meant the class of real-valued functions defined on $\mathbb{R}$ whose restriction to any bounded interval I is in $H^k(I)$. An element g in H^k_{loc} is periodic of period L if $g(x) \equiv g(x+L)$, and the collection of all such g is herein denoted H^k_L. If g is in H^k_L, then $\partial^j g(x) \equiv \partial^j g(x+L)$ for $0 \leqslant j < k$, and conversely, an element f in $H^k([0, L])$ which satisfies these k periodicity conditions has a unique extension to an element of H^k_L. The norm on H^k_L is the norm of $H^k([0, L])$. Let $\mathscr{V}^k_T = C(0, T; H^k_L)$ and more generally $\mathscr{V}^{k,l}_T = C^l(0, T; H^k_L)$.

Proposition 16. Let $g \in H^k_L$ where $k \geqslant 1$ and let $T > 0$. Then there exists a solution u in $\mathscr{V}^{k,\infty}_T$ to the periodic initial-value problem for (1.1). Furthermore, $u \in \mathscr{V}^1_\infty$ and for $j \geqslant 1$, $\partial^j_t u \in \mathscr{V}^{k+1,\infty}_T$.

The proof of proposition 16 proceeds exactly as in Benjamin *et al.* (1972, § 3) supplemented by our remarks in § 3. The Green function for $I - \partial^2_x$ on H^k_L is the same as the Green function on H^k, and hence the periodic initial-value problem can be recast as an integral equation

$$u(x, t) = g(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y) \{u(y, \tau) + \tfrac{1}{2}u^2(y, \tau)\} \, dy \, d\tau, \tag{B 2}$$

where K is as defined in (3.4) and (B 2) is viewed as an operator equation in $\mathscr{V}^k_T$.

The method of proof is to demonstrate existence of a solution in $\mathscr{V}^1_{t_0}$ for t_0 sufficiently small by means of a contraction mapping argument. This solution in the small is then extended to a global solution in $\mathscr{V}^1_\infty$ by use of the integral invariant

$$E(u) = \int_0^L [u^2(x, t) + u^2_x(x, t)] \, dx = E(g) \tag{B 3}$$

for all $t \geqslant 0$.

Finally this weak solution is shown to be an element of $\mathscr{V}^{k,\infty}_T$ by bootstrap arguments as given in Benjamin *et al.* (1972, § 3) and in the present § 3.

With proposition 16 in hand, define a regularized periodic initial-value problem, just as before.

$$u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0, \quad u(x, 0) = g(x), \tag{B 4}$$

where g is specified in H^k_L. The transformation (3.2) sends u to a solution v of the periodic initial-value problem, with period $M = \epsilon^{-\frac{1}{2}} L$,

$$v_t + v_x + vv_x - v_{xxt} = 0, \quad v(x, 0) = \epsilon g(\epsilon^{\frac{1}{2}} x), \tag{B 5}$$

where the initial data in (B 5) is a member of H^k_M. Existence of solutions for the problem (B 5) is concluded by appeal to proposition 16. By following the inverse transformation from (B 5) to (B 4), existence of solutions u in $\mathscr{V}^k_T$ to the periodic initial-value problem (B 4) is obtained.

Now the plan is, as before, to derive *a priori* bounds for smooth (i.e. H^∞_L) solutions of the regularized problem (B 4). This proceeds exactly as in § 4 with no changes worthy of comment. Then suppose g is specified in H^k_L, $k \geqslant 1$. Regularize g by a multiplication operation on its Fourier coefficients. If $g \sim \Sigma_k a_k e^{2\pi i kx/L}$, then define

$$g_\epsilon(x) \sim \Sigma_k \varphi(\epsilon^{\frac{1}{6}} k) \, a_k \, e^{2\pi i kx/L}, \tag{B 6}$$

in analogy with (5.1). g_ϵ is in H^∞_L. A periodic version of lemma 5 holds for the $\{g_\epsilon\}$ of course, and hence the solutions u_ϵ of (B 4) posed with initial data g_ϵ are smooth and satisfy estimates as in corollaries 1 and 2 to lemma 5. We may then pass to the limit as $\epsilon \downarrow 0$, either weak-star in $L^\infty(0, T; H^k_L)$ as in appendix A (only the argument is easier this time since the underlying spatial domain $[0, L]$ is bounded) or strongly in $\mathscr{V}^k_T$ as in § 5 and, for the case $k = 2$, appendix A.

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A solution u is obtained to the periodic initial-value problem (B 4) for g. Moreover, u depends continuously in $\mathscr{V}^k_T$ on g in H^k_L, as one sees by mimicking the developments in § 6. We have outlined the proof of the following result.

THEOREM 10. Let $g \in H^k_L$, where k is a positive integer. Then there exists a solution u, which is in $L^\infty(0, T; H^k_L)$, for all $T > 0$, to the periodic initial-value problem (B 4) for g. If $k \geqslant 2$, then u is unique and $u \in \mathscr{V}^k_\infty$. Furthermore, if $k \geqslant 2$ and T is fixed, u depends continuously in $\mathscr{V}^k_{T'}$ on g in H^k_L.

The further developments of § 8 may now be carried over for the periodic initial-value problem with no essential change in detail.

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